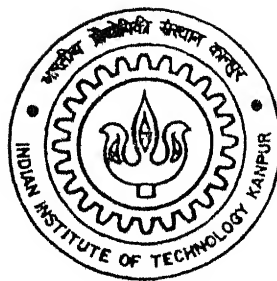


YIELD MANAGEMENT: MODELLING, ANALYSIS AND IMPLICATIONS

by

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Indian Institute of Technology Kanpur

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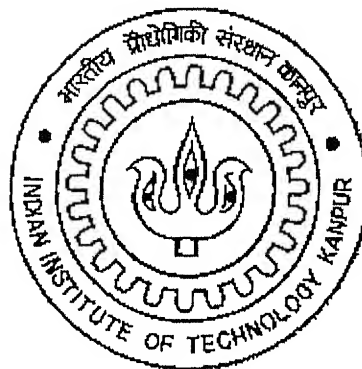
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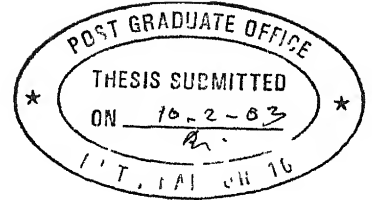
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CERTIFICATE



It is certified that the work contained in the thesis entitled, "*Yield Management-Modelling, Analysis, and Implications*" by Sabale Anand H. has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

A handwritten signature in cursive script, which appears to read "Sanjeey Swami".

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ABSTRACT

The term *yield management* is used in many service industries like, airline, hotel industry, car rental business, to describe techniques to allocate limited resources among a variety of customers. By adjusting this allocation a firm can optimize the total revenue with given capacity constraints. These techniques are often used by firms with perishable goods or services that cannot be stored. Hence, these concepts and tools are also called *perishable asset revenue management* (Weatherford and Bodily, 1992).

The objective of this thesis is to propose demand estimation models and suggest optimal pricing strategy in the context of yield management. We propose six different theoretical modeling approaches for demand estimation. Demand is assumed to be function of price, time and inventory remaining. These models are –intrinsically linear model, Swami and Khairnar (2003) model, Tretheway and Weinberg(1991) model, van Ryzin (1994) model, a neural network approach for demand estimation and nonhomogeneous Poisson demand process (Leemis, 1991). Four representative demand data sets for car rental business are used (Wang, 2001). For empirical analysis, data is simulated with variations at 5%, 10% and 20% of demand.

We also propose an approach to determine optimal pricing policy for revenue maximization when the demand is stochastic assuming Markovian transitions. We use Markov Decision Process (Puterman, 1994) modelling approach and show the optimal pricing strategy for a small numerical example. The managerial implications of yield management are discussed. The directions for future research explore possibilities for further work in this area.

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CHAPTER 1

Introduction

Yield management is the term to describe techniques to allocate limited resources among a variety of customers. By adjusting this allocation a firm can optimize the total revenue or yield on the investment in capacity. These techniques are often used by the firms with perishable goods or by firms with services that cannot be stored at all. Hence, these concepts and tools are also called PARM (Perishable Asset Revenue Management) (Weatherford and Bodily, 1992). PARM is a systematic approach to maximize profits in an environment where the variable costs are relatively lower than fixed costs. In these situations profit maximization is usually accomplished by maximizing revenue, subject to satisfying the capacity constraints.

Business environments with certain characteristics are more appropriate for the practice of yield management. Yield management is applicable when it is expensive or impossible to store excess resources, for example, in hotel industry, we cannot store tonight's room for use by tomorrow's customers. The firms having variable costs of the product much lower than fixed costs are suitable candidates for the application of yield management. Similarly, if the cost of adding capacity in a business is very high and has a long lead time it is more suitable for yield management. This is the case with airlines where cost of adding a new aircraft is quite high. For hotel industry, the time required to add to room capacity is much longer.

Yield management is appropriate when the product or service is perishable that is there is a point after which the product is either not available or no longer saleable. After departure of a flight, empty seats are no longer salable. The firm may apply yield management when it can differentiate among customer segments, and each segment has a different demand curve. Purchase restrictions and refundability requirements help in segmenting market between leisure and business customers for airline as well as hotel industry. When the product can be sold in advance, yield management is applicable. Nowadays, airlines provide centralized reservations for seats (Belobaba 1987). Techniques of yield management are applicable for the seasonal products like fashion goods and products, which have consistent demand pattern. Yield management can help smoothen the demand curve by stimulating demand during low demand times and increasing revenue during high demand times.

Yield Management in Practice

American Airlines were the pioneers in yield management. In the 1960's, they developed the first on-line reservation system. Later on, they became so specialized that they formed SABRE (Semi-Automated Business Research Environment) group. This group deals with centralizing and controlling reservation activity. The SABRE is American's "store front," its interface with reservation agents and, hence, with customers. Through computer terminals in airports and travel agencies, SABRE makes available a complex array of itinerary and fare options. SABRE also controls the availability of seat inventory, and it is a key component of the yield management system because it permits collecting the data on which models like the overbooking system depend (Paul Davis, 1994).

Robert Cross, President of Aeronomics, a leading commercial provider of yield management systems for the airline industry, estimated that on average, airlines increase their revenue 2% after employing yield management systems. American Airlines reported that the benefits of yield management were US\$1.4 billion over 3 years, approximately 4 to 5% of revenue (Thomas Cook, 1998). Likewise, Hertz car rentals reports a 5% increase in average revenue per rental with its yield management system (Wirtz 2001). A 1997 study by Coopers & Lybrand showed that revenue management accounted for nearly one-fifth of the hotel industry's 6.3% increase in average daily rate in 1996.

Yield management system at American Airlines generates almost \$1 billion in annual incremental revenue. The techniques that American Airlines now uses for yield management are the result of an evolutionary process that began in the early 1970s. With such laudable results in the airline and hotel industries, it is not surprising that the practice of yield management is rapidly catching the interest of other service industries such as shipping, performing arts, media and broadcasting services, professional services, car rental and even hospital services.

Decisions in Yield Management

The sellers would like to sell their products to high value customers having a high valuation so that higher margins can be achieved. At the same time, if they wait too long for those customers to appear, they might end the selling period with unsold units that could have been sold to low value customers. For this trade off to be nontrivial, demands estimation is necessary. For perishable products such as airline tickets and fashion goods, different

prices are widely used by the sellers. The price option is often rendered through the differentiation of the time when a good is purchased and by the amount of unsold inventory firms may have on hand. Price decisions on perishable products are affected by the length of time remaining before products are spoiled and by the levels of the unsold inventory.

Another important decision in yield management is regarding overbooking, when there is a chance that a customer may not appear (Netessine and Shumsky 2002). For example, it is possible for a customer to book a ticket on an airline flight and not show up for the departure. In this case the airline may end up flying an empty seat resulting in loss of revenue to the company. In order to account for such no-shows, overbooking decisions are required. Overbooking decision is based on historical data but if by chance, a larger than usual portion of the customers shows up, then certain customers end up getting not served. Hotels, car rental agencies, some restaurants and even some non emergency health care providers routinely overbook.

Management Objectives in Yield Management

Possible management objectives include maximization of profit or contribution. Contribution towards fixed cost is defined as revenue minus variable cost, while for profit we must subtract fixed costs from the contribution and accounts for taxes also. Maximum capacity utilization goal focuses on selling every available unit. Sales people use this approach if they are rewarded according to the number of units sold. With the objective of maximization of revenue, the cost side is ignored, probably because variable costs are negligible. Ideal goal for management in yield management would be to extract each customer's maximum price, but it is not possible because it would involve a tough negotiation with every customer.

Constraints in Yield Management

The possible constraints faced by yield manager include operational constraints, such as, how to allocate fixed capacity of units. For example, in the airlines, the number of seats available is fixed. Other operational constraints in airlines may be scheduling of routes, number of planes and frequency of flights.

In marketing constraints, there is a minimum tolerable customer service level. The level of customer service can be measured in a variety of ways, such as, customers who are

not served because of overbooking. A strategic aim for top level management may be to become the price leader in the market (Kraft and Tretheway 1986). This would impose strategic constraints on the yield manager.

Costs in Yield Management

Two kinds of costs are associated with yield management. First the variable costs for the unit of product or service and the costs associated with the event for which a customer is denied a reservation. The latter is more difficult to measure because lost goodwill cannot be observed directly (Kraft and Tretheway 1986)

Demand Modelling in Yield Management

In general, in yield management, booking limit decision is just one aspect of a more general business question - *How should a firm market and distribute goods to multiple customer segments?* To answer this question, a firm must use tools for pricing and demand estimation (Belobaba 1989). Good modeling and forecasting of demand are key factors for pricing decisions. Pricing policies are fundamental component of the daily operations of manufacturing and service industries. The reason is that price is one of the most effective variables that managers can manipulate to encourage or discourage demand in short run. Price is not only important from a financial point of view but also from an operational standpoint as a tool that helps to regulate inventory pressures. Airline companies and retail chains are good examples of industries where dynamic pricing policies are becoming key drivers of the company's performance (Bitran and Mondschein 1997).

In literature, demand is considered as a function of price only. Exponential demand models are commonly used to model demand in retail sector. These models are of the form $G(p) = \exp(-n \cdot p)$ where n is measure of demand elasticity per unit of price. Other models using constant elasticity have also been proposed (Bitran 1995). Karlin and Carr (1962) have considered several types of demand functions as $p^{-a} \exp(-b \cdot p)$ for $a, b \geq 0$. Gallego and van Ryzin (1994) has proposed demand as $Demand = a \cdot \exp(-b \cdot p)$.

We propose to decompose the demand into a set of three factors namely price effect, time of purchase and influence of product inventory remaining. Our first model, intrinsically linear model considers the multiplicative effect of these three factors. The

model is intrinsically linear as logarithms on both sides convert multiplicative function into a linear one. We propose use of nonhomogeneous Poisson process (NHPP) (Leemis 1991) to model arrival of customers at service center. Neural networks with their ability to derive meaning from complicated or imprecise data can be used to extract patterns and detect trends that are too complex to be noticed. A neural network approach for demand estimation is proposed. We have used Swami and Khairnar (2003) model, Tretheway and Weinberg (1991) model and van Ryzin (1994) model are used as benchmark models. Intrinsically linear model performs better for the given data set which shown by comparative analysis of all the models. Neural network approach also gives impressive results, however, neural networks do not have sufficient explanatory power (Law 1999)

Pricing Policies in Yield Management

Pricing policy models proposed in literature can be broadly classified into deterministic and stochastic models. Deterministic models assume that the seller has perfect information about the demand process. This is a major simplification and is appropriate where demand is highly predictable at the beginning of the season. This is found in some cases of new products and fashion goods. Deterministic models are easy to analyze and they provide a good approximation for the more complicated but realistic stochastic models (Gallego and van Ryzin 1994). Pricing models with stochastic demand are more appropriate to describe real life situations where the demands are unpredictable over time and managers are forced to react dynamically by adjusting prices, as uncertainty reveals itself (Alstrup et.al. 1986). We propose to use stochastic demand for pricing policies. We tackle this problem by using stochastic dynamic programming techniques, also known as Markov Decision Processes (MDP) (Puterman 1994). We consider a long term, discrete time, and finite horizon decision problem. At every decision point during selling horizon, the manager collects all relevant information about the inventory positions and sales and establishes the prices at which the products should be sold. We propose pricing model under Markovian assumptions on the demand process.

To summarize, our objective in this thesis is to propose various demand estimation models in the context of yield management. By comparative analysis, we intend to specify the "best" model that works best according to chosen criteria and given data sets. In the

second part of the thesis, we propose pricing policy using MDP approach, which is more appropriate for real life applications.

The rest of the thesis is organized as follows. Chapter 2 deals with literature review of yield management. In Chapter 3, we propose different demand estimation models. We then provide the empirical analysis of the demand models in Chapter 4. Chapter 5 deals with pricing policies. We then summarize the results and also discuss managerial implications. We conclude by pointing out limitations of the current research and directions for future research of the proposed methodology.

CHAPTER 2

Literature Review

2.1 Yield Management- Taxonomy and Research Overview

Kraft and Tretheway (1986) describe the modern concept of airline seat management. Seat management is a system whereby airlines determine when the seats might be flown empty, whether to offer those seats at discount, and how many seats to offer, at what level of discount and what restrictions to attach to such seats. The objective is to maximize profits for a given flight. Basic seat management procedures are presented. The paper concludes with a discussion of criteria of choosing alternative seat management systems.

A comprehensive literature review of yield management appears in Weatherford and Bodily (1992). The authors propose the term PARM (Perishable Asset Revenue Management) to denote the field that combines the areas of yield management, overbooking and pricing of perishable assets. They identify the common characteristics in which yield management is applicable as discussed in Chapter 1. The paper also gives a 14 element comprehensive taxonomy of the PARM. Table 2.1 shows these elements with their descriptors. We provide the following descriptions:

- Nature of resources—As a PARM element, nature of resources means the nature of units of perishable assets. The units can be discrete like airline seats or continuous like electric power.
- Capacity—In case of yield management, capacity is considered to be fixed, but it is not necessary.
- Prices—In a typical PARM situation, prices are predetermined by a pricing group that is separate from any decisions about the number of discounts units to sell.
- Diversion—It means the customers who buy at full price are separate from the ones who buy at discount rates.
- Displacement—These are the bumping procedures which happen when demand exceeds capacity.

Authors have identified other elements such as willingness to pay, discount price classes, reservation demand, show up of discount reservations, show up of full price reservations, group reservations, asset control mechanism and decision rules.

Weatherford and Bodily (1992) propose different decision rules that might be used for two price class problem. These decision rules are as follows-

- a. Pick a fixed allocation q^* prior to reserving the first customer, knowing that it will not be changed later for any reason.
- b. Pick a fixed time t^* prior to reserving the first customer and accept all requests prior to t^* .
- c. Select a (q^*, t^*) decision rule prior for the first reservation, or monitor everything continuously and decide when to curtail reservations.

This paper also gives currently solved problems in yield management and problems are grouped under the headings like simple generic allocation problems, optimally set pricing, stochastic discount demand, overbooking, problems with diversion, problems with displacement effects and dynamic problems.

Elements	Descriptors
A Resource	Discrete/ Continuous
B Capacity	Fixed/ Nonfixed
C Prices	Predetermined/ Set Optimally/ Set jointly
D Willingness to pay	Buildup/ Drawdown
E Discount Price Classes	1/ 2/ 3/...../ In
F Reservation Demand	Deterministic/ Mixed/ Random— independent / Random—correlated
G Show—up of Discount Reservation	Certain/ Uncertain without cancellation/ Uncertain with cancellation
H Show—up of Full—Price Reservation	Certain/ Uncertain without cancellation/ Uncertain with cancellation
I Group Reservation	No/ Yes
J Diversion	No/ Yes
K Displacement	No/ Yes
L Bumping Procedure	None/ Full—Price / Discount/ FCFS/ Auction
M Asset Control Mechanism	Distinct/ Nested
N Decision Rule	Simple Static/ Advanced Static/ Dynamic

Table 2.1 Comprehensive Taxonomy of PARM Problems

(Source: Weatherford and Bodily 1992)

2.2 Marginal Revenue and Optimal Booking Models

2.2.1 Littlewood (1972) studies a two class, single leg problem and proposes a marginal seat revenue model. He proposed that an airline should continue to sell discount seats as long as the following condition is satisfied:

$$\rho_B \geq \rho_Y \Pr[Y > k - n] \quad (2.2.1.1)$$

where ρ_B is average revenue from discount passengers, ρ_Y is average revenue from full fare passengers, $\Pr[\cdot]$ denotes probability, Y is full fare demand, k is number of seats available for two fare classes and n is the number of discount seats sold. In simple terms, the above formula specifies – *sell an additional discount seat as long as the discount revenue equals or exceeds the expected full fare revenue from the seat* Bhatia and Parikh (1973) develop a continuous version of the Littlewood's rule. Ritcher (1982) provides a marginal analysis which proves that Littlewood's rule gives an optimal allocation.

2.2.2 Belobaba (1987, 1989) proposes a probabilistic model and extends the marginal seat revenue principle to the expected marginal seat revenue rule (EMSR). EMSR decision model takes into account the uncertainty associated with estimates of future demand as well as the nested structure of booking limits in airline reservation systems. Let S_i is the number of seats allocated to a particular fare class, then cumulative probability that all requests for a fare class will be accepted as a continuous function of S_i ,

$$P_i(S_i) = P[r_i \leq S_i] = \int_0^{S_i} P_i(r_i) dr_i \quad (2.2.1.2)$$

Conversely,

$$P[r_i \leq S_i] = \int_0^{S_i} P_i(r_i) dr_i = 1 - P_i(S_i) = P_i(S_i) = \text{probability of spill occurring} \quad (2.2.1.3)$$

then expected marginal seat revenue of S_i^{th} seat j fare class i is,

$$EMSR_i(S_i) = f_i P_i(S_i)$$

where

f_i – average fare level for class i .

EMSR model developed takes into account the uncertainty associated with estimates of future demand as well as nested structure of booking limits in airline reservation systems. The EMSR rule seeks to revise the static estimation of optimal booking limits continuously when additional information about booking and acceptance of seats becomes available.

2.2.3 Brumelle and McGill (1993) address the problem of determining optimal booking policies for multiple fare classes that share the same seating pool when the seats are booked in nested fashion. This paper shows that a fixed limit booking policy that maximizes expected revenue can be characterized by a simple set of conditions. This paper allows demand distributions to be either discrete or continuous. The connection of seat allocation problem to the theory of optimal stopping is demonstrated as –

Optimal protection levels $P_1^*, P_2^*..$ must satisfy the condition

$$\delta_+ ER_K[P_K^*] \leq f_{k+1} \leq \delta_- ER_K[P_K^*] \text{ for each } k=1,2$$

where $ER_K[P_K]$ is the expected revenue from K highest fare classes when P_K seats are protected for those classes and δ_+ and δ_- denote the right and left derivatives with respect to P_K . This paper also shows that EMRS method can both over and underestimate the optimal protection levels by constructing a counter-example

2.3 Dynamic and Stochastic Models in Yield Management

2.3.1 Stochastic discount demand is another stream of research in which the demands of the full fare class and discounted class are assumed to be dependent. Brumelle et al. (1990) deal with the problem of allocating airline seats between two nested fare classes when the demands are stochastically dependent. This paper presents a model for seat allocation that allows for demand dependency between fare classes.

$$n^* = \max \left\{ 0 \leq n \leq k : \Pr[Y \geq kn \mid B \geq n] \leq \frac{\rho_B}{\rho_Y} \right\} \quad (2.3.1.1)$$

Considering goodwill premium,

$$n^* = \max \left\{ 0 \leq n \leq k : \Pr[Y \geq kn \mid B \geq n] \leq \frac{\rho_B}{\rho_Y + \rho_g} \right\} \quad (2.3.1.2)$$

Considering upgrading probability,

$$n^* = \max \left\{ 0 \leq n \leq k : \Pr[Y \geq kn \mid B \geq n] \leq \frac{\rho_B - \gamma(\rho_Y + \rho_g)}{\rho_Y + \rho_g} \right\} \quad (2.3.1.3)$$

where

Y is the full fare demand

k is the number of seats available

n is the number of discount tickets sold

γ is the upgrade probability

ρ_B is the average revenue per discount

ρ_Y is the average revenue per full fare booking

ρ_g is the goodwill cost per rejected full fare passenger.

The extension of the model includes the full fare passenger spillage and the impact of seat allocation decisions on passenger goodwill.

2.3.2 In stochastic dynamic programming approach, Alstrup et al. (1986) treat the airline booking process as a Markovian nonhomogeneous sequential decision process. They present a model for a fixed non stop flight with two types of passengers. The model considers cancellations, reservations prior to departure, no shows, denied boarding and downgrading of passengers. This model is solved by two dimensional stochastic dynamic programming. The objective achieved is to minimize the difference between the maximal gain obtainable and actual gain.

The model is –

$$V(BC, BM, O) = \sum_{i=-\infty}^{BC} \sum_{j=-\infty}^{BM} PNC(i, BC) - PNM(j, BM) - COST(BC, BM, j) \quad (2.3.2.1)$$

where,

$V(BC, BM, O)$ – expected total costs at departure

BC, BM – number of passengers already booked on C and M class, where C and M represent two different class of customers

$PNM(j, BM)$ – demand for M class passengers

$PNC(j, BC)$ – demand for C class passengers

$COST(BC, i, BM-j)$ – total cost.

2.3.3 Bitran and Mondschein (1995) have studied optimal strategies for renting hotel rooms when there is a stochastic and dynamic arrival of customers. These customers are assumed to be from different segments. Authors formulate the problem as a stochastic and dynamic programming model. They characterize the optimal policies as functions of the capacity and the time left until the end of planning horizon. There is no assumption regarding the particular order between the arrivals of the different classes of customers. The model allows multiple types of rooms, downgrading and requests for multiple nights. Authors propose the optimal policies for the single night case as—*given a period of time, if a request is accepted for any certain capacity, then it is also accepted for any larger capacity*

2.3.4 The finite horizon stochastic knapsack is useful for optimizing sales of perishable commodities. Slyke and Young (1998) deal with this problem. In the proposed model, K types of customers arrive stochastically. Customer type k , has an integer weight w_k , a value b_k , and arrival rate λ_k . They consider the analogy of a continuous time horizon to a knapsack with capacity W . The decision rule is of the form: *for each arrival that fits in the remaining capacity either accept it while receiving b_k and giving up capacity or reject it while not losing capacity*. The problem is modeled as continuous time, discrete state, and finite horizon dynamic programming problem.

2.3.5 Gallego and van Ryzin (1994) investigate the problem of dynamically pricing perishable inventories when demand is price sensitive and stochastic, and the firm's objective is to maximize expected revenues. They formulate this problem using inventory control, and obtain structural monotonicity results for the optimal intensity as a function of stock level and the length of horizon. For a particular family of exponential family of demand functions, such as $a \cdot \exp(-ap)$, where a is any constant and p is price, optimal pricing policy in closed form is presented. The model extensions include demand is compound poisson, finite numbers of prices are allowed, demand rate is time varying, overbooking and random cancellations.

2.4 Pricing Models

2.4.1 Optimal policies of yield management with multiple predetermined prices are dealt by Feng and Xiao (1998). Their model assumes that the products are offered at multiple predetermined prices over time, demand is price sensitive and obeys Poisson process, and price is allowed to change monotonically. The authors propose that, to maximize the expected revenue, management needs to determine the optimal times to switch between prices based on the remaining season and inventory. Major results of the work include the exact solution for the continuous time model. Given the state of time to go and on hand inventory, management can handily determine the optimal price by inspecting those previously calculated thresholds.

2.4.2 Talluri and van Ryzin (2001) analyze the revenue management under a general discrete choice model of consumer behavior. The authors deal with an airline yield management problem on a single flight leg in which the buyers' choice of fare classes is modeled explicitly. The problem is to find, at each point of time, the optimal subset of fare classes to offer. The analysis also provides insights into the nested allocation policies.

2.4.3 Pricing for seasonal products is dealt by Bitran et al. (1997). The authors study pricing policies when selling seasonal products in retail stores. A continuous time problem is considered where a seller faces a stochastic arrival of customers. Arrival of potential customers is assumed to be a Poisson process.

The problem is formulated as

$$V_t(c) = \max \{ M_t(1 - F_t(p) * (p_t * V_{t+1}(c-1))) + (1 - M_t(1 - F_t(p))V_{t-1}) \} \quad (2.4.3.1)$$

Boundary Conditions

$V_t(0) = 0$ and $V_0(c) = 0$ for all values of t and c .

and

$$m_t = \int_{\tau=t}^{t+\Delta t} \lambda \tau d\tau$$

where,

λ_t – Customers arrival rate

c – Total inventory at the beginning

L – Length of planning horizon

m_t - demand for seasonal product

$f_t(x)$ – probability density function of reservation price at t

$F_t(x)$ – cumulative distribution function of reservation price at t

$V_t(c)$ – maximum expected profit.

The model is solved by one-dimensional nonlinear optimization problem backward in time. Fibonacci algorithm is used to solve the nonlinear problem.

2.4.4 Shugan et al. (1998) deals with pricing strategies implemented in yield management such as early discounting, overbooking and limiting early sales. This paper has two objectives-to seek a conceptual foundation for the strategic pricing of capacity constrained services and to arrive at precise conditions when specific strategies such as early discounting and limiting sales are best. Their analysis suggests that yield management works best when price-insensitive customers prefer to buy later than price-sensitive consumers. The authors suggest following policy – *with only one period of service and when cancellations are known with certainty, over-book by exactly the expected number of no-shows As that number increases, overbooking should increase and charge lower average prices and sell more tickets to the discount price segment.*

2.5 NHPP Models

Customer arrivals for the purchase of perishable product or service are a time dependent process because of dynamic effects present in the yield management. Kao and Chang (1988) consider the use of nonhomogeneous Poisson process (NHPP) in modeling time dependent arrivals to service organizations. The authors propose the use of a piecewise polynomial to represent the rate function. They present two maximum likelihood estimators for estimating the parameters of the piecewise polynomial function. One is based on arrival times and the other is based on aggregated counts. They use a procedure based on thinning for generating arrival times from such process In the same stream of research a nonparametric technique for estimating the cumulative intensity function of a NHPP is developed by Leemis (1991). This technique does not require any arbitrary parameters from the modeler.

2.6 Miscellaneous Approaches

2.6.1 Deterministic Network Approach

Glover et al (1982) use a deterministic network model to solve a multiple fare class problem. They formulate the problem with special side constraints. A network optimizing component finds that flow on each arc which maximizes revenue on the carrier network without violating the aircraft capacity constraints and the upper bounds posed by the demand forecasts. This model was implemented by Frontier Airlines.

2.6.2 Linear Programming Approach

Brumelle et al. (1995) investigate the monopoly pricing problem for perishable product using linear programming approach. The model incorporates the use of artificial restrictions. This paper deals with number of price levels and then shows that for n levels of restriction, at most $n+1$ price levels are needed. Pricing problem is formulated as linear programming problem.

2.6.3 Graph Theory Approach

Diaz et al. (1997) use a graph theory approach for allocating seats and setting optimal prices in an origin—destination network. Inputs for the model include demand forecasts, computer reservation system restrictions and aircraft capacities. The concept of a split graph is used to derive cutting plane. The model proposes to include all fare classes in given origin destination (OD) route. The problem is to select a price structure and allocate seats to each fare class in the price structure such that total expected revenue is maximized for the entire OD network. The solution procedure consists of eliminating variables with low booking probabilities, then eliminate a single price structure from the first group of constraints to attain a full dimensional polyhedral, and then initiate a branch and bound procedure

2.6.4 Neural Network Approach

Wang (2001) proposes a hybrid model to solve yield management problems. According to this approach, data that reflect the relationships between reservations and price as well as time are used to locate the high dimensional threshold band with monotonic relationship streamlines

The data on these streamlines are used to train the back propagation neural network. The trained neural network represents the monotonic streamlines of the threshold band. These streamlines are used to estimate the probability of cumulative reservations at a given price on a certain day. The dynamic programming approach is then applied to find the optimal pricing scheme based on the threshold band composed of the streamlines

2.6.5 Product Diffusion Model

Diffusion models are widely used to model evolution of demand. In this framework, a population of consumers of size N gradually purchases the product. The rate at which consumers buy the product depends linearly on the number of previous purchases and the fraction of innovators existing in the population. Innovators are those customers who buy the product independent of the other consumers' action. In Bass's diffusion model (1969), the rate of purchase at time t is given by

$$\frac{dD(t)}{dt} = pN + (q - p)D(t) - \frac{q}{N} D^2(t) \quad (2.6.5.1)$$

where p is the fraction of innovators and q is a measure of diffusion effect. The combination of this diffusion model with scarcity effect is proposed by Swami and Khairnar (2003) model. This work provides a model for diffusion of products, which are available in limited quantity with known expiration date. The model is developed on the foundation of classical Bass (1963) model and the psychological effect known as scarcity principle, which states that *the opportunities seem more valuable to us when they are less available* (Cialdini 1985). Building on this insight, the model is developed on two cornerstones of scarcity principle, limited availability of stock and time deadline. The authors suggest that their basic model can be manipulated such that the sales depend on both cumulative adoption (x) and time (t) at any instance, that is,

$$y = A + Bx + Cx^2 + Dx^3 + Et + Ftx + Gtx^2 + Htx^3 \quad (2.6.5.2)$$

where y is the cumulative sales, $A-H$ are coefficients which are functions of original fixed parameters of the problem such as length of planning horizon, homogenous potential consumer population size and capacity. Other functional forms of the model are as follows

- i. $y = A + Bx + Cx^2$
- ii. $y = A + Bx + Cx^2 + Dx^3$
- iii. $y = A + Bx + Cx^2 + Dt$

$$\text{iv} \quad y = A + Bx + Cx^2 + Dtx$$

$$\text{v.} \quad y = A + Bx + Cx^2 + Dtx^2$$

Their results show that scarcity has a significant effect on the diffusion patterns, when the product availability is limited. The authors propose to use the model for forecasting of the products whose availability is limited.

2.7 Yield Management for Nonprofit Sector

Yield management techniques have been highly effective for many profit organizations. Yield management is intended to maximize profits for a capacity constrained service by price discriminating among customers. Metters and Vargas (1999) have extended the yield management for nonprofit sector, where profit maximization is no longer a goal. A general heuristic is presented to assist decision makers in pricing decisions. The technique is demonstrated at a nonprofit child care center that provides discounts to low income families. Two systems have been presented that extend concepts from single leg yield management models to the nonprofit environment—an algorithm to assist in pricing decisions that recognizes the basic business trade offs and a linear programming based yield management system to allocate capacity, once prices are set.

CHAPTER 3

Demand Models for Yield Management

Proper modeling and forecasting of demand are key for effective yield management procedures. The usual demand models consider the set of potential customers and divide them into different families, each one having its own set of attributes including needs, budgets and quality expectations. Depending on the factors such as price and other attributes such as capacity, potential customers will decide whether or not to purchase the product. Using van Ryzin's (1993) terminology, potential customers are divided into shoppers-those customers that search for products but do not buy because of price or quality considerations, and buyers-those customers that are effectively willing to buy a product. In most applications the seller is only capable of collecting information about the set of buyers according to sales data

3.1 Intrinsically Linear Demand Model

The simplest approach is to decompose the demand into a set of different factors each one addressing a specific aspect of the problem. Demand is decomposed into three parts-

1. Price elasticity effect
2. Estimate of demand as a function of time
3. Influence of available inventory on customers purchasing behavior.

3.1.1 Price Elasticity of Demand

Price elasticity of demand measures the responsiveness of demand to a given change in price (Kotler 2000) and is found using the equation:

$$PED = \text{Percentage change in quantity demanded} / \text{Percentage change in price}$$

Figure (3.1.1.1) and Figure (3.1 1.2) show graphically price elasticity of demand

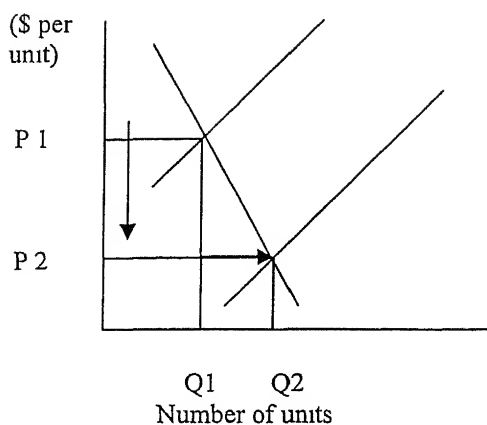


Figure (3.1.1 1) Low Price Elasticity

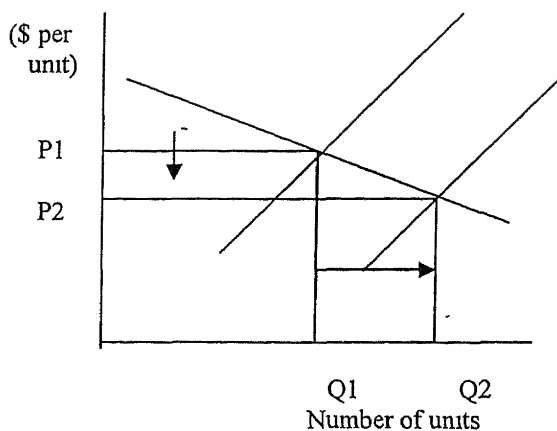


Figure (3.1.1.2). High Price Elasticity

In Figure 3.1.1.1, a large change in price per unit leads to a small change in demand in terms of number of units and in Figure 3.1.1.2, a small change in price leads to large change demand. Thus each price change leads to a different level of demand. The relationship between alternative prices and the resulting current demand is captured in a demand curve. In normal case demand and price are inversely related: the higher the price, the lower the demand and vice versa. The first step in estimating demand is to understand what affects price sensitivity. Nagle (1992) has identified nine factors-

1. Unique Value Effect-Buyers are less price sensitive when the product is more distinctive.
2. Substitute-Awareness Effect - Buyers are less price sensitive when they are less aware of substitutes.
3. Difficult Comparison Effect-Buyers are less price sensitive when they cannot easily compare the quality of the substitutes.
4. Total Expenditure Effect-Buyers are less price sensitive the lower the expenditure is as a part of their total income
5. End Benefit Effect-Buyers are less price sensitive the smaller the expenditure is to the total cost of the end product.
6. Shared Cost Effect-Buyers are less price sensitive when the part of the cost is borne by another party.

7. Sunk Investment Effect-Buyers are less price sensitive when the product is used is used in conjunction with assets previously bought
8. Price Quality Effect-Buyers are less price sensitive when the product is assumed to have more quality.
9. Prestige or Exclusiveness, Inventory Effect-Buyers are less price sensitive when they cannot store the product.

If demand hardly changes with a large change in price, we say the demand is inelastic. If demand changes considerably by small changes in the price then the demand is said to be elastic. If demand is elastic, seller will consider lowering the price. A lower price will produce more total revenue. This makes sense as long as variable costs do not increase disproportionately, which the case in yield management. Price elasticity depends on the magnitude and direction of the price change. It may be negligible with a small price change and substantial with a large price change

3.1.2 Estimate of Demand as Function of Time

Modeling of estimate of the market size as a function of time depends on the seasonality of demand and the life cycle of the product. Diffusion models are widely used to model this evolution of demand. In this framework, a population of consumers gradually purchases the product. The rate at which consumers buy the product depends linearly on the number of previous purchases and the fraction of innovators existing in the population. Innovators are those customers that buy the product independent of the other consumers' action Figure 3.1 2 1 shows communication influences in Bass diffusion model.

The Bass model derives from a hazard function, which is defined as the probability that an adoption will occur at time t given that it has not yet occurred. In the context of marketing, it is the probability that a customer will adopt the product if he/ she has not adopted yet. The rate of purchase at time t is given by

$$\frac{dD(t)}{dt} = pN + (q - p)D(t) - \frac{q}{N} D^2(t) \quad (3.1.2.1)$$

where p is the fraction of innovators and q is a measure of diffusion effect.

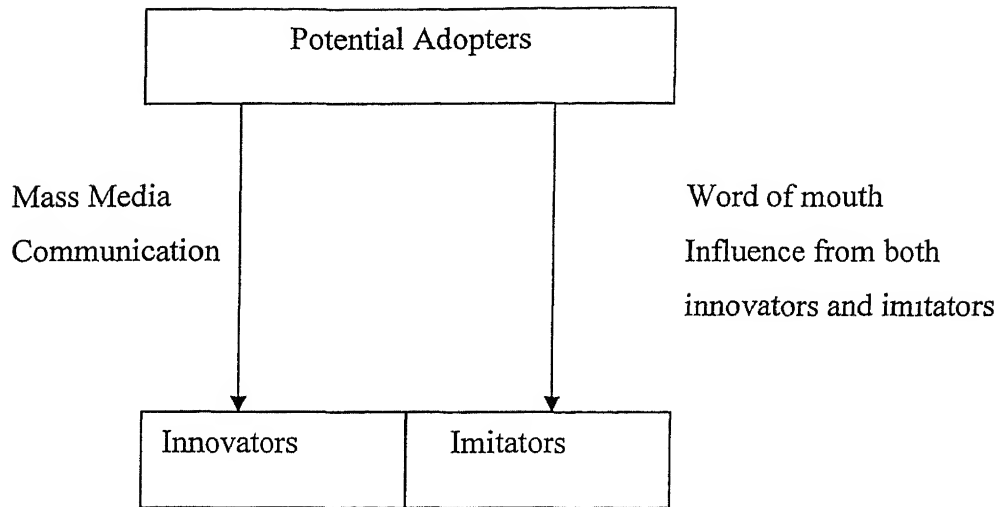


Figure (3.1.2.1). Communication Influence in Bass Diffusion Model

3.1.3 Influence of Inventory Remaining

Influence of available inventory on customer purchase behavior is well explained by scarcity principle. According to this principle, opportunities seem more valuable to us when they are less available and vice versa (Cialdini 1985). Social psychologists have found that as a weapon of influence, the scarcity effect has notable power in directing human action. The scarcity principle trades on the consumers' weakness for heuristics or short cuts. It has been found that the things that are difficult to get are typically evaluate better than those that are easy to get. The consumers often use an item's availability to help them decide on quality. The two cornerstones of the scarcity principle are limited number and deadline approaches. In limited numbers approach, a customer is informed that a certain product is in short supply. The usual intent is to convince customers of an item's scarcity. The effect of stock remaining and potential customers on likelihood of customers is presented in Figure (3.1.3.1). Scarcity effect is observed when there is a large potential customer and goods remaining are low.

		Potential customers remaining	
		HI	LO
Stock remaining	HI	No Effect	Decrease
	LO	Increase	No Effect

Figure (3.1 3.1). Effect of the Stock Remaining and Potential Customers on Likelihood of Adoption

Tretheway and Weinberg (1991) propose a model which considers the effect of price and inventory remaining. Their model is as follows

$$D = L * X * e^{-\alpha * p} \quad (3.1.3.1)$$

On the other hand, van Ryzin et al (1994) propose a model which only considers the effect of price on demand. Their proposed model is as follows

$$D = \alpha * e^{-\beta * p} \quad (3.1.3.2)$$

We propose to use scarcity principle, along with effects of price and time, in our model. In yield management problem there is capacity constraint and at the end of selling horizon, scarcity effect could be visible. In other words, consumers may have higher propensity to buy perishable products, if fewer of them are left, or near time deadline, than otherwise.

Thus we propose,

Demand = f (Price elasticity effect, Time, Remaining Inventory)

$Demand = f(p, t, X)$

$$D = e^{-a * p} * t^b X^c \quad (3.1.3.3)$$

where,

$e^{-a * p}$ captures the price elasticity. t^b represents the time effect on demand. This may model seasonality of the demand, and X^c models the influence of the available inventory on customer's purchase behavior

This model considers the multiplicative effect of three factors and is intrinsically linear by logarithmic transformation on both sides of Equation 3.1.3.3.

Assumptions for model

It is assumed that the seller has monopolistic market power over the set of buyers. Competition might be present in this formulation, but it is hidden. We do not consider any strategic behavior from customer side. Similarly, customers are assumed to be price takers, meaning they observe the price list offered by the seller and react by buying or not buying some of the products.

3.2 Swami and Khairnar (2003) Model

The combination of diffusion model with scarcity effect is proposed in Swami and Khairnar (2003) model. This work provides a model for diffusion of products, which are available in limited quantity and with known expiration date. The model is developed on the foundation of classical Bass (1963) model and the psychological effect known as scarcity principle (Cialdini 1985).

The model suggests that sales depend on both cumulative adoption (x) and time (t) at any instance, that is

$$y = A + Bx + Cx^2 + Dx^3 + Et + Ftx + Gtx^2 + Htx^3 \quad (3.3.1)$$

where A-h are coefficients which are functions of original fixed parameters of the problem

Other functional forms of the model are as follows

- i $y = A + Bx + Cx^2$
- ii. $y = A + Bx + Cx^2 + Dx^3$
- iii $y = A + Bx + Cx^2 + Dt$
- iv. $y = A + Bx + Cx^2 + Dtx$
- v. $y = A + Bx + Cx^2 + Dtx^2$

This approach does not consider effect of price on sales.

3.3 Nonhomogeneous Poisson Process Demand Model

Early description of statistical models of passenger booking and no-show behaviors are found in Beckman (1958). In this paper, the authors compare Poisson, negative binomial and gamma models of passenger arrivals. Lyle (1970) has modeled demand as composed of a gamma systematic component with Poisson random errors. We assume that total number of customers arriving at a car rental center follow a Poisson process. The stochastic process $\{N(t), t \geq 0\}$ is said to be Poisson process if-

- 1. Customers arrive one at a time
2. The number of arrivals in the time interval $(t, t + s)$ is independent of $\{N(u), 0 \leq u \leq t\}$
3. The distribution of $\{N(t + s) - N(t)\}$ is independent of t for all $t, \geq 0$

where $N(t)$ represents number of customers arrived at time t .

In case of car rental business, we can assume that customers arrive at a time. But Assumptions 2 and 3 might be questionable in car rental business. Assumption 2 states that the number of arrivals in the interval $[t, t+s]$ is independent of the number of arrivals earlier time interval $[t, 0]$. In reality this assumption may not hold because number of customers arrived earlier has effect on the next arrival of customers. This can be related to the imitation effect presented in the Bass model. A large number of arrivals in period $[t, 0]$ might affect arrivals between $[t,s]$. Assumption 3 states that the inter arrival rate is independent of time. Scarcity principle might also play an important role in customer arrival. Thus while modeling customer arrival process, assumption of Poisson process may not be sufficient. The arrival of customers could be dependent on time as well as on the number of arrivals in previous period. We propose to model the customer arrival as nonhomogenous Poisson process (NHPP), (Leemis L. 1991) where customer arrival varies over time.

Let $\lambda(t)$ be the arrival rate of customer arrival at time t . If customers arrive at the system in accordance with Poisson process with constant rate λ , then $\lambda(t) = \text{constant}$ for all $t \geq 0$. However in the NHPP, $\lambda(t)$ is a function of time t . If the arrival rate $\lambda(t)$ varies with time, then the inter-arrival times are not identically distributed, thus it is not appropriate to fit a single distribution for inter-arrival times.

Let $\theta(t) = E[N(t)]$ be the expectation function or cumulative intensity function, for all $t \geq 0$. If $\theta(t)$ is differentiable for a particular value of t , we formally define $\lambda(t)$, intensity function, as

$$\lambda(t) = \frac{d}{dt} \theta(t) \quad (3.4.1)$$

Intuitively, $\lambda(t)$ will be large in inter-arrivals for which the expected number of arrivals is large. The following theorem shows that the number of arrivals in the interval $(t, t + s]$ for a NHPP is a Poisson random variable whose parameter depends on both t and s

Theorem

If $\{N(t), t \geq 0\}$ is a NHPP with continuous expectation function $\theta(t)$ then,

$$P[N(t+s) - N(t) = k] = \frac{e^{-b(t,s)} * [b(t,s)]^k}{k!} \quad (3.4.2)$$

for $k = 0, 1, 2, \dots$

where

$$b(t,s) = \theta(t+s) - \theta(t) = \int_t^{t+s} \lambda(t) dt \quad (3.4.3)$$

(Law and Kelton 2000)

A nonparametric method for estimating the rate function is proposed in Leemis (1990) as follows-A NHPP is a generalization of an ordinary Poisson process where events occur randomly over time at the rate of $\lambda(t)$ events per unit time. The rate at which events occur in a NHPP varies over time as determined by intensity function, $\lambda(t)$ The cumulative intensity function is defined by

$$\theta(t) = \int_0^t \lambda(t) dt \quad t > 0 \quad (3.4.4)$$

and is interpreted as the expected number of events by time t . The probability of exactly n events occurring in the interval $(a, b]$ and is given by

$$P(k = n) = \frac{\left\{ \left[\int_t^{t+s} \lambda(t) dt \right]^n \right\} * \left\{ e^{-\int_t^{t+s} \lambda(t) dt} \right\}}{n!} \quad (3.4.5)$$

$n = 0, 1, 2, \dots$

Thus probability distribution of customer demand can be found by using Equation 3.4.5.

3.4 Neural Network Approach

3.4.1 Foundation

An Artificial Neural Network (ANN) is an information processing paradigm that is inspired by the way biological nervous systems, such as the brain, process information. The key element of this paradigm is the novel structure of the information processing system. It is composed of a large number of highly interconnected processing elements (neurons) working in unison to solve specific problems. ANNs learn by example and are configured for a specific application, such as pattern recognition or data classification, through a learning process (Zurada 2000).

Neural networks, with their remarkable ability to derive meaning from complicated or imprecise data, can be used to extract patterns and detect trends that are too complex to be noticed by either humans or other computer techniques. A trained neural network can be thought of as an "expert" in the category of information it has been given to analyze. This expert can then be used to provide projections given new situations of interest and answer "what if" questions. ANN has adaptive learning capability that is an ability to learn how-to-do-tasks based on the data given for training or initial experience. An ANN can create its own organization or representation of the information it receives during learning time.

Neural networks take a different approach to problem solving than that of conventional computers. Conventional computers use an algorithmic approach that is the computer follows a set of instructions in order to solve a problem. Unless the specific steps that the computer needs to follow are known, the computer cannot solve the problem. This restricts the problem solving capability of conventional computers to problems that we already understand and know how to solve. Neural networks process information in a similar way the human brain does. The network is composed of a large number of highly interconnected processing elements (neurons) working in parallel to solve a specific problem. Since neural networks learn by examples, they cannot be programmed to perform

a specific task. The examples must be selected carefully otherwise useful time is wasted or even worse the network might be functioning incorrectly. The disadvantage is that because the network finds out how to solve the problem by itself, its operation can be unpredictable.

An artificial neuron is a device with many inputs and one output. A simple neuron is shown in Figure 3.4.1. The neuron has two modes of operation; the training mode and the using mode. In the training mode, the neuron can be trained to fire (or not), for particular input patterns. In the using mode, when a taught input pattern is detected at the input, its associated output becomes the current output. If the input pattern does not belong in the taught list of input patterns, the firing rule is used to determine whether to fire or not.

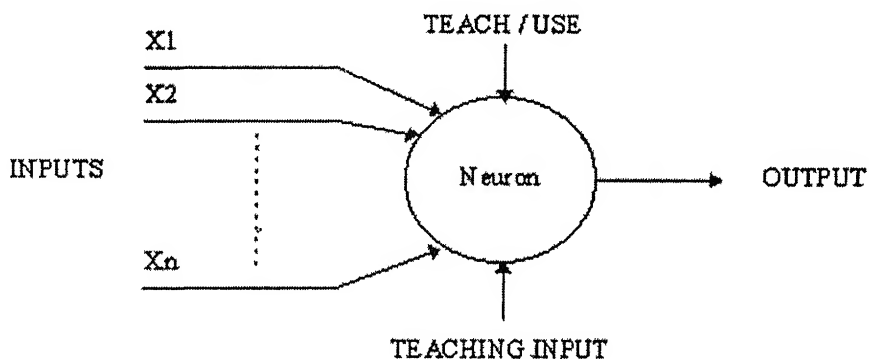


Figure (3.4.1): A simple neuron

The firing rule is an important concept in neural networks and accounts for their high flexibility. A firing rule determines how one calculates whether a neuron should fire for any input pattern. It relates to all the input patterns, not only the ones on which the node was trained.

Feedback networks (Figure 3.4.2) can have signals traveling in both directions by introducing loops in the network. Feedback networks are very powerful but can get extremely complicated. Feedback networks are dynamic; their 'state' is changing continuously until they reach an equilibrium point. They remain at the equilibrium point until the input changes and a new equilibrium needs to be found. Feedback architectures are also referred to as interactive or recurrent.

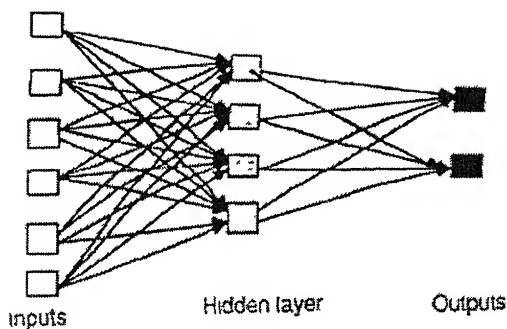


Figure (3 4.2). An Example of a Simple Feedback Network

3.4.2 Network Layers

The most common type of neural network consists of three groups, or layers, of units. A layer of "input" units is connected to a layer of "hidden" units, which is connected to a layer of "output" units. The activity of the input units represents the raw information that is fed into the network. The activity of each hidden unit is determined by the activities of the input units and the weights on the connections between the input and the hidden units. The behavior of the output units depends on the activity of the hidden units and the weights between the hidden and output units.

3.4.3 Learning Process

All learning methods used for adaptive neural network can be classified into two categories. The first is supervised learning, which incorporates an external teacher, so that each output unit is told what its desired response to input signals ought to be. During the learning process global information may be required. Examples of supervised learning include error correction learning, reinforcement learning and stochastic learning. An important issue concerning supervised learning is the problem of error convergence, that is, the minimization of error between the desired and computed unit values. Unsupervised learning uses no external teacher and is based upon only local information.

3.4.4 Transfer function

The behavior of neural network depends on both the weights and the input output function (transfer function) that is specified for the units. This function falls into one of the three categories: linear, threshold or sigmoid. For linear inputs, the output activity is proportional to the total weighted output. For threshold units, the output at one of two levels depends on

whether the total input is greater than or less than some threshold value. For sigmoid units, the output varies continuously but not linearly as the input changes. Sigmoid units bear a greater resemblance to real neurons than do other two

3.4.5 Back Propagation Algorithm

In order to train a neural network to perform some task, we must adjust the weights of each unit in such a way that the error between the desired output and the actual output is reduced. This process requires that the neural network computes the error derivatives of the weights (EW). In other words it must calculate how the error changes as each weight is increased or decreased slightly

The back propagation algorithm is the most widely used method for determining the EW. The back-propagation algorithm is easiest to understand if all the units in the network are linear. The algorithm computes each EW by first computing the actual error, the rate at which the error changes as the activity level of a unit is changed. For output units, the actual error is simply the difference between the actual and the desired output. To compute the actual error for a hidden unit in the layer just before the output layer, we first identify all the weights between that hidden unit and the output units to which it is connected. We then multiply those weights by the actual errors of those output units and add the products. This sum equals the actual error for the chosen layer, we can compute in like fashion the actual errors for other layers, moving from layer to layer in a direction opposite to the way activities propagate through the network. Once the actual error has been computed for a unit, it is straight forward to compute the EW for each incoming connection of the unit. The EW is the product of the actual error and the activity through the incoming connection. The mathematical approach of Back Propagation Algorithm is given in Appendix B.

We propose to use BPLMS, that is, back propagation least mean square error algorithm. The BPLMS learning algorithm is an iterative gradient algorithm designed to minimize the mean square error between the actual output and the desired output by modifying network weights. A neural network is actually a mapping function representing the relationship between its inputs and outputs. A neural network with three input nodes, a single hidden layer and a single output node is used in this study. The training set represents a vector (price, time, and on hand inventory).

The standard BPLMS neural networks learning algorithm, however, has difficulty in generating an effective interpolation. That is, given a set of training data for the algorithm, the final result could not be predictable (Kawabata 1991). Thus to get desired results based on a limited number of training points one has to use additional information and knowledge. The basic disadvantage of the BPLMS is that it must be imposed with monotonicity constraints during the training process. When the training data carry enormous statistical fluctuations, the monotonic relationships between a data point and its adjacent data point may not be maintained.

CHAPTER 4

Empirical Analysis

4.1 Data Description

In this section we present the results of the different demand models proposed earlier. In order to test these models we have used the data of a car rental business (Wang 2001) Data consist of number of reservations made before a deadline of 14 days A typical data set is presented in Table 4 1

We have used four representative data sets There are five price levels namely \$19, \$24, \$29, \$34, and \$39 Against each price level, there are a number of reservations made. These data correspond to 14 days before reservation. In a car rental business, these five price levels are interpreted as the cost of different car rental services. A car rental business provides various kinds of vehicles from taxi to luxury cars according to the needs of customers A typical car rental business manages fleet of vehicles starting from a four wheeler to a minibus The pricing decision is influenced by the competition in the market. Data provides the pattern of the reservations made for these vehicles with different prices Car rental business shares the characteristics similar to a yield management problem. Important characteristics are - the product is perishable, that is, there is a point after which product is no longer available. Secondly, the product can be sold in advance Demand for the product is seasonal and the sale is constrained by capacity The demand process in case of car rental business possesses the similarities that match well with the modeling environment that we have proposed in earlier sections The sales of car for rent are affected by price offered for the product, timing of sale and the previous sale.

In context of the sales of the car rental business, we test six models. Tretheway and Weinberg (1991) model, van Ryzin (1994) model and Swami and Khairnar (2003) model are used as the benchmark models. Tretheway and Weinberg (1991) model considers the effect of price and inventory remaining. The model is given as Equation (3.1 3.1). van Ryzin (1994) model assumes demand to be function of price and it is represented in Equation (3 1.3 2) Swami and Khairnar (2003) model as presented in Equation 3.3 1, proposes demand in terms of previous cumulative sales and time. All the three benchmark models propose demand in terms of one or more of the factors. price, time and inventory remaining However, none of the benchmark models considers effect of the three factors

simultaneously. We propose to consider the effect of three factors in a single model. We present three approaches to model the demand as function of three factors. First we propose an intrinsically linear demand model to consider three factors in multiplicative way. This model is analyzed by using linear regression analysis. We then propose use of NHPP (Leemis 1991) to model the arrival of customers at service center. Finally a neural network approach to estimate demand is proposed. The results of the proposed models and benchmark models are presented in Table 4.4 1

4.2 Simulation Setup

We simulate the various patterns of demand in order to account for the different kinds of variations that may arise in these kinds of situations. To generate these patterns, we add error term to the four original data sets. The error is assumed to follow the normal distribution with mean 0 and variance σ^2 . Thus $\varepsilon \sim N(0, \sigma^2)$ represents the error function. Using the different values for the variance we generate different demand patterns. Thus if x is the number of reservations made before certain day and y is the generated cumulative demand pattern, we represent relation between them in the form,

$$y = x + \varepsilon \quad (4.2.1)$$

The various scenarios examined are aimed at replicating what if situations such as what if the demand at each point was different from the actual values given. We use three levels of variance, namely low, medium and high. The values of the standard deviation used for different levels are - low variation 5%; medium variation 10%; high variation 20% of demand. We have used the variance that represent the maximum deviation from the basic pattern, lower as well as upper. Simulated data sets are presented in the Appendix A. Data sets are designated using three digits, such as, 1 1.1, where first digit indicates data set number between 1 to 4, second digit indicates variation, 1 for low variation, 2 for medium variation, 3 for high variation and last digit indicates replication number between 1 to 4. In simulated data sets, some observations are negative. We assume zero demand instead of the negative values.

4.3 Estimation Method

The estimation procedures for the benchmark models and proposed models are presented as follows.

4.3.1 Benchmark Model Analysis

We propose to use three benchmark models. These models are presented below

Tretheway and Weinberg (1991) model

$$D = L * X * e^{-\alpha * p} \quad (4.3.1.1)$$

van Ryzin (1994) model

$$D = \alpha * e^{-\beta * p} \quad (4.3.1.2)$$

Swami and Khairnar (2003) model

$$y = A + Bx + Cx^2 + Dtx \quad (4.3.1.3)$$

All the three model are analyzed by using regression analysis Both Tretheway and Weinberg (1991) and van Ryzin (1994) models are intrinsically linear models The logarithmic transformation on both sides leads to a linear model. Swami and Khairnar (2003) propose different variations for their model. We have selected the above form of the model because it this form considers the multiplicative effect of time and remaining inventory. We use SPSS (Statistical Package for Social Sciences) for regression analysis The results of the benchmark models are discussed in the next section.

4.3.2 Proposed Models Analysis

4.3.2.1 Intrinsically Linear Model

We propose intrinsically linear model which considers the effect of price, time and remaining inventory in a multiplicative way. The model is as follows

$$D = e^{-a * p} * t^b X^c \quad (4.3.2.1.1)$$

We estimate the parameters of the proposed model using regression analysis.

4.3.2.2 NHPP Approach

We propose to use NHPP approach to model demand estimation for a car rental business data set. In NHPP, customer arrival is a time dependent process In our modelling environment, customers arriving at car rental service center is a time dependent process as well as it also depends on the price also. We consider the effect of price in NHPP model.

The parameter of NHPP is the arrival rate $\lambda(t, p)$, which in our model is function of both price and time as shown in Equation 4.3 2 2.1

$$\lambda(t) = a + bt + D_1 p_1 + D_2 p_2 + D_3 p_3 + D_4 p_4 \quad (4.3.2.2.1)$$

where a , and b are regression coefficients and D_1, D_4 are dummy variables.

Leemis (1991) suggests a procedure to estimate the NHPP parameter. We use regression analysis to estimate the parameter to consider the effect of price and time. In regression analysis, inter-arrival rate is dependent variable and time and price are used as independent variables. This arrival rate function is used in equation 3.4 5 to determine the probability of arrival

$$P(k = n) = \frac{\left\{ \left[\int_t^{t+s} \lambda(t) dt \right]^n \right\} * \left\{ e^{-\int_t^{t+s} \lambda(t) dt} \right\}}{n!} \quad (4.3.2.1.3)$$

We propose that estimated demand is that value of n for which $P(k = n)$ is maximum. In other words, estimated demand represents the number of customers with maximum probability of arrival for given price and time. The mean absolute deviation (MAD) is used as the performance criteria. We propose to predict the minimum number of customers for given price and time with certain 90% probability. For this purpose, the individual probabilities for each number of customer arrivals are added cumulatively. When this cumulative probability adds to 10%, we say that *with 90% confidence, the least demand will be say n , for a given price and time*

4.3.2.3 Neural Network Approach

We propose third approach for demand estimation by using neural network approach. Neural networks are trained with the input data. Input data consists of price, time and cumulative sales one period earlier. Output data or target values are the actual demands. Neural networks are trained by using MATLAB software. The MATLAB code used for the training purpose is given in Appendix D.

4.4 Estimation Results

The comparative results of demand estimation by proposed and benchmark models are tabulated in Table 4.4.1. Neural network approach is tested for original data sets only. Parameter values of the proposed and benchmark models are presented in Table 4.4.2 to

Table 4.4.6. All models are tested using mean absolute deviation (MAD). Comparative analysis of benchmark models and proposed models using t test is shown in Table 4.4.7 to Table 4.4.10

- Intrinsically linear model has good fit for the demand data indicated by the large R^2 values and statistically significant estimates of the model. The t - statistics and p - level is used as an indicator of the statistical significance.
- For higher variations the proposed model performs better compared to benchmark models. As the level of variation increases, the R^2 values for the benchmark models reduce. This would be observed because for higher variations, demand patterns fluctuations are more.
- Swami and Khairnar (2003) model has R^2 values comparable to that of proposed model for original data sets. The explanation for good fit by Swami and Khairnar (2003) model is that the data used for analysis has consistent pattern. The variations in the data set are low. Swami and Khairnar (2003) model has less significant results for higher variations.
- Tretheway and Weinberg (1991) and van Ryzin (1994) model have higher MADs and these values increase as the variation in demand data set increases. In our data set, there are predefined five price levels. van Ryzin (1994) considers effect of price only, which explains the reason for low R^2 values.
- Estimates of demand by neural network approach are significant, shown by minimum MAD. The use of back propagation algorithm provides good estimates that are difficult to simulate using the traditional mathematical functions. The results obtained by using neural network are very impressive. However, neural networks do not structure knowledge with symbols like mathematical functions (Law 1999).

Table 4.4.7 Estimation Results for Neural Network Model

Data Set	Mean Absolute Deviation
Data Set 1	1.48
Data Set2	1.36
Data Set 3	1.42
Data Set4	1.6

- NHPP approach results are less significant when compared to other benchmark and proposed models. Parameter estimation of the NHPP is crucial for better results. We propose to consider effects of price along with time factor. Better results can be achieved with better parameters for NHPP. In sum, the proposed models, namely, Intrinsically Linear Model and Neural Network approach provide good fit for the demand patterns which are characterized by the effect of price, time and influence of remaining inventory.
- Mean absolute deviations (MAD) of benchmark models and proposed models are presented in Table 4.4.7 to Table 4.4.10. MAD of two models is tested by F-test for variance analysis. All models have different variances for MAD because F statistics is higher than the critical value. We then analyze the mean of MAD for different models using t- test for unequal variances. Results of t- test show that MAD of Intrinsically Linear Model is less compared to that of benchmark models This leads us to conclusion that proposed model performance is better

4.5 Predictive Validity Test

In this thesis we propose different demand estimation models which consider the effect of price, time and influence of remaining inventory. In the result section we showed that the proposed models have good fit. However, the relative analysis of the models would be more complete with an assessment of the predictive validity test of the models.

In this section, we present predictive validity test. We use two methods for predictive validity. In the first method, inter-group method, we use the three data sets to estimate the model parameters and predict for the fourth data set. We then calculate the mean absolute deviation (MAD) for the models. Intra-group method uses the data of the same set. We use data for four price levels and predict the demand for the fifth price level. Similarly in the intra group method, we use data set for seven days and predict for the eighth day data. MAD is used as performance criteria.

The results of predictive validity test are presented in Table 4.4 11. The MAD values for predictive validity test are computed for original and simulated data sets The average values of MADs are reported in Table 4.4.11. Intrinsically linear model performs consistently good for original as well as for simulated data sets Swami and Khairnar (2003) model gives comparable results for original data sets but results for medium and

high variation simulated data sets are not good. Original data sets are of consistent pattern and variation is low. This may explain variation in results for Swami and Khairnar (2003) model Tretheway and Weinberg (1991) and van Ryzan (1994) models MAD varies between 2.2 to 2.7. Both the models do not consider multiplicative effect of time and inventory remaining NHPP results show constant MAD for all methods, close to 30. The results of NHPP model largely depends on the parameter of the model. Results show consistent lag for predicted demand values

Neural network results can not analyzed like other models because neural networks do not structure knowledge into mathematical functions, although results of neural network are impressive.

Table 4.1 Original Data Sets

	Days before reservation	0	2	4	6	8	10	12	14
	Price								
Data Set 1	19	48	45	40	32	28	25	18	10
	24	44	42	38	34	25	22	17	11
	29	40	34	30	27	22	17	12	9
	34	36	30	28	23	19	13	10	8
	39	28	23	19	14	10	8	6	2

Data Set 2	19	46	44	41	36	25	22	19	12
	24	44	40	37	33	27	20	16	10
	29	41	35	30	28	21	18	12	10
	34	35	30	25	20	17	13	9	7
	39	24	21	17	12	9	4	3	2

Data Set3	19	49	46	40	37	30	26	21	17
	24	45	41	37	34	26	24	16	10
	29	42	37	31	28	26	17	11	10
	34	33	28	23	18	14	11	9	6
	39	30	24	20	16	11	8	4	3

Data Set 4	19	49	47	43	40	32	24	20	18
	24	42	40	35	32	28	22	14	9
	29	39	36	30	25	20	16	10	8
	34	34	29	24	19	10	10	9	7
	39	25	20	16	13	8	6	3	2

Table 4.4.1

**Comparative Results of Demand Estimation by
Proposed and Benchmark Models (Original and
Simulated Data Sets)**

Model	Intrincially Linear		Swami &		Tretheway and		van Ryzin		NHPP
	Model		Khairnar Model		Weinberg		Model		model
	R sqr	MAD	R sqr	MAD	R sqr	MAD	R sqr	MAD	MAD
Data set									
1	0.96	1.68	0.95	1.79	0.92	2.37	0.87	2.27	25.3
1.1.1	0.95	1.62	0.95	1.87	0.89	2.32	0.85	2.25	26.32
1.1.2	0.96	1.52	0.94	2.32	0.84	2.45	0.83	2.28	26.61
1.1.3	0.94	1.73	0.95	2.43	0.86	2.28	0.82	2.27	29.86
1.1.4	0.92	1.69	0.96	1.79	0.88	2.39	0.87	2.31	26.34
1.2.1	0.92	1.75	0.91	2.27	0.87	2.26	0.81	2.29	30.20
1.2.2	0.93	1.79	0.92	2.39	0.84	2.27	0.79	2.26	25.22
1.2.3	0.91	1.79	0.84	2.24	0.86	2.41	0.78	2.31	29.16
1.2.4	0.89	1.80	0.87	2.29	0.83	2.47	0.79	2.27	28.31
1.3.1	0.9	1.68	0.9	2.37	0.87	2.29	0.8	2.32	32.70
1.3.2	0.84	1.73	0.89	2.43	0.81	2.43	0.82	2.27	31.40
1.3.3	0.87	1.78	0.9	2.36	0.86	2.37	0.81	2.34	32.90
1.3.4	0.86	1.70	0.84	2.41	0.79	2.52	0.86	2.28	30.32
2	0.95	1.70	0.96	1.90	0.95	2.03	0.84	2.72	27.1432
2.1.1	0.93	1.72	0.92	2.02	0.96	2.16	0.82	2.73	29.9521
2.1.2	0.95	1.93	0.89	1.81	0.82	2.43	0.86	2.74	29.6325
2.1.3	0.91	1.99	0.88	2.09	0.86	1.99	0.84	2.64	26.1407
2.1.4	0.97	1.67	0.9	1.93	0.84	2.34	0.82	2.60	26.3533
2.2.1	0.89	1.83	90	2.07	0.84	2.20	0.8	2.42	27.3431
2.2.2	0.85	1.82	0.84	2.04	0.85	2.37	0.82	2.47	30.9721
2.2.3	0.83	1.65	0.86	1.86	0.87	2.39	0.78	2.58	25.8499
2.2.4	0.88	1.93	0.79	2.01	0.79	2.40	0.81	2.50	31.2468
2.3.1	0.82	1.97	0.81	2.01	0.76	2.49	0.74	2.61	26.9915
2.3.2	0.81	1.83	0.81	1.83	0.72	2.53	0.75	2.62	29.8683
2.3.3	0.86	1.92	0.79	1.80	0.78	2.20	0.79	2.53	25.1499
2.3.4	0.82	1.83	0.83	1.99	0.8	2.58	0.76	2.78	25.12

Table 4.4.1

**Comparative Results of Demand Estimation by
Proposed and Benchmark Models (Original and
Simulated Data Sets)**

Model	Intrincially Linear Model		Swami & Khairnar Model		Tretheway and Weinberg		van Ryzin Model		NHPP model
	R sqr	MAD	R sqr	MAD	R sqr	MAD	R sqr	MAD	
Data set									
3	0.96	1.92	0.95	2	0.96	2.14	0.925	1.825	29.897
3.1.1	0.94	2.085	0.96	1.8179	0.94	2.2111	0.87	1.95	29.3027
3.1.2	0.92	1.9211	0.93	1.8152	0.89	2.2052	0.85	2.073	31.2045
3.1.3	0.96	1.8245	0.95	1.926	0.92	2.2027	0.83	2.085	31.6685
3.1.4	0.94	2.0456	0.96	2.0912	0.88	2.1359	0.82	1.925	29.6913
3.2.1	0.9	2.0119	0.91	1.8873	0.85	2.09	0.87	2.157	30.1108
3.2.2	0.9	1.9288	0.89	2.0712	0.84	2.248	0.81	1.999	29.6896
3.2.3	0.91	1.9937	0.87	2.0604	0.86	2.2428	0.79	2.018	29.0376
3.2.4	0.89	1.938	0.85	2.0977	0.83	2.137	0.81	1.892	30.4505
3.3.1	0.84	2.0546	0.89	2.0897	0.85	2.0904	0.82	2.089	30.2069
3.3.2	0.87	1.9822	0.84	2.0463	0.86	2.1331	0.75	1.9	31.3562
3.3.3	0.87	1.8635	0.9	1.8452	0.81	2.2383	0.78	1.89	29.9451
3.3.4	0.86	1.914	0.84	2.0408	0.82	2.0212	0.74	1.965	30.12
4	0.96	2.07	0.96	1.86	0.95	2.28	0.928	2.033	28.3313
4.1.1	0.93	1.979	0.95	1.9121	0.94	2.1582	0.9	2.285	29.9598
4.1.2	0.92	2.0457	0.94	2.2044	0.89	2.2219	0.85	2.178	31.782
4.1.3	0.89	1.9774	0.96	2.0305	0.91	2.1843	0.89	2.278	29.3011
4.1.4	0.95	2.0917	0.96	2.0984	0.87	2.2734	0.84	2.111	28.4649
4.2.1	0.92	1.9818	0.89	2.0904	0.85	2.1915	0.83	2.14	31.777
4.2.2	0.85	1.9047	0.87	2.025	0.84	2.2142	0.81	2.019	29.5295
4.2.3	0.89	1.9544	0.9	1.9974	0.87	2.19	0.8	2.179	28.9068
4.2.4	0.88	1.832	0.87	2.0285	0.82	2.2807	0.81	2.183	31.7672
4.3.1	0.84	2.0989	0.85	2.0593	0.81	2.1795	0.76	2.148	29.5193
4.3.2	0.81	1.8337	0.89	2.2884	0.72	2.2698	0.79	2.085	31.331
4.3.3	0.86	1.8929	0.85	1.9533	0.78	2.2659	0.8	2.149	28.486
4.3.4	0.82	1.8397	0.84	2.1513	0.8	2.2201	0.79	2.268	29.15

Table 4.4.2

Estimation Results of Intrinsically Linear Model (Original Data Sets)

	Data Set 1			Data Set 2		
	Parameter Value	t-stat	Significance	Parameter Value	t-stat	Significance
constant	1 44	5 93	0.00	1 49	2 97	0.01
a	-0 07	-9.10	0.00	-0 02	-3.33	0.00
b	0 85	-5 00	0 00	0 54	8.96	0 00
c	0 10	4 58	0.00	0 09	0 98	0 33
F	206 75		0.00	169 20		0 00

Data Set 3			Data Set 4			
	Parameter Value	t-stat	Significance	Parameter Value	t-stat	Significance
constant	1 86	4.01	0 00	2.06	3.83	0.00
a	-0 02	-4.05	0.00	-0.03	-4 06	0.00
b	0 47	8.31	0.00	0.47	7.47	0.00
c	0 03	0.41	0.69	-0.01	-0.16	0.88
F	200.97		0.00	194.05		0

Table 4.4.3

Estimation Results of Swami Khairnar Model (Original Data Set)

Original Data set 1				Original Data set 2			
	Parameter value	t-stat	Significance	Parameter value	t-stat	Significance	
a	8.17	8.48	0.00	6.82	7.16	0.00	
b	0.21	6.43	0.00	0.25	7.28	0.00	
c	0.00	-0.47	0.65	0.00	-1.52	0.14	
d	0.02	4.94	0.00	0.02	4.69	0.00	
F	234.81			238.46			

Original Data set 3				Original Data set 4			
	Parameter value	t-stat	Significance	Parameter value	t-stat	Significance	
a	7.61	7.26	0.00	5.95	6.15	0.00	
b	0.27	8.28	0.00	0.29	9.77	0.00	
c	0.00	-2.54	0.02	0.00	-3.24	0.00	
d	0.01	3.96	0.00	0.02	5.04	0.00	
F	198.98			242.62			

Table 4.4.4 Estimation Results of Tretheway and Weinberg (1991) Model
Original Data Set

Data Set 1				Data Set 2			
	Parameter value	t-stat	Significance	Parameter value	t-stat	Significance	
L	1.55	1.79	0.08	1.89	10.93	0.00	
a	-0.45	-5.49	0.00	-0.02	-5.38	0.00	
X	0.57	15.99	0.00	0.48	19.72	0.00	
F	199.08		0.00	314.29		0.00	

Data Set 3				Data Set 4			
	Parameter value	t-stat	Significance	Parameter value	t-stat	Significance	
L	1.95	13.39	0.00	1.89	10.91	0.00	
a	-0.02	-5.94	0.00	-0.02	-5.64	0.00	
X	0.45	21.79	0.00	0.48	19.99	0.00	
F	380.13		0.00	35.38		0.00	

Table 4.4.5 **Estimation Results of van Ryzin (1994) Model**
Original Data Set

	Data Set 1			Data Set 2		
	Parameter value	t-stat	Significance	Parameter value	t-stat	Significance
a	4.33	12.04	0.00	4.37	10.42	0.00
b	-0.05	-3.87	0.03	-0.05	-3.57	0.04
F	14.99		0.00	52.14		0.00

	Data Set 3			Data Set 4		
	Parameter value	t-stat	Significance	Parameter value	t-stat	Significance
a	4.57	23.05	0.00	4.67	27.21	0.00
b	-0.05	-6.97	0.01	-0.04	-7.05	0.01
F	62.21		0.00	45.23		0.00

**Table 4.4.7 Comparative Results for MAD for
Intrinsically Linear Model and Swami and
Khairnar (2003) Model**

F-Test Two-Sample for Variances

	<i>Swami and Khairnar</i>	
	<i>Model</i>	<i>Intrinsically Linear Model</i>
Mean	2 228054727	1 711493177
Variance	0 059251204	0 006101305
Observations	13	13
df	12	12
F	9 711234151	
P(F<=f) one-tail	0 000201923	
F Critical one-tail	2 686633138	

t-Test Two-Sample Assuming Unequal Variances

	<i>Swami and Khairnar</i>	
	<i>Model</i>	<i>Intrinsically Linear Model</i>
Mean	2.228054727	1 711493177
Variance	0 059251204	0 006101305
Observations	13	13
Hypothesized Mean	0	
df	14	
t Stat	7 285554633	
P(T<=t) one-tail	1 99888E-06	
t Critical one-tail	1.76130925	
P(T<=t) two-tail	3.99776E-06	
t Critical two-tail	2.144788596	

**Table 4.4.8 Comparative Results for MAD for
Intrinsically Linear Model and Trethway
and Weinberg Model**

F-Test Two-Sample for Variances

	<i>Trethway and Weinberg Model</i>	<i>Intrinsically Linear Model</i>
Mean	2 370234312	1 711493177
Variance	0 007082126	0 006101305
Observations	13	13
df	12	12
F	1 160755856	
P(F<=f) one-tail	0 400223992	
F Critical one-tail	2 686633138	

t-Test Two-Sample Assuming Unequal Variances

	<i>Trethway and Weinberg Model</i>	<i>Intrinsically Linear Model</i>
Mean	2 370234312	1 711493177
Variance	0 007082126	0 006101305
Observations	13	13
Hypothesized Mean	0	
df	24	
t Stat	20.68579582	
P(T<=t) one-tail	4 16104E-17	
t Critical one-tail	1.710882316	
P(T<=t) two-tail	8 32207E-17	
t Critical two-tail	2.063898137	

Table 4.4.9 Comparative Results for MAD for Intrinsically

Linear Model and van Ryzan (1994) Model

F-Test Two-Sample for Variances

	<i>van Ryzan Model</i>	<i>Intrinsically Linear Model</i>
Mean	2 286790861	1 711493177
Variance	0 000757437	0 006101305
Observations	13	13
df	12	12
F	0 124143477	
P(F<=f) one-tail	0 000510889	
F Critical one-tail	0 372212483	

t-Test: Two-Sample Assuming Unequal Variances

	<i>van Ryzan Model</i>	<i>Intrinsically Linear Model</i>
Mean	2 286790861	1 711493177
Variance	0 000757437	0 006101305
Observations	13	13
Hypothesized Mean	0	
df	15	
t Stat	25 04621204	
P(T<=t) one-tail	5.92569E-14	
t Critical one-tail	1 753051038	
P(T<=t) two-tail	1.18514E-13	
t Critical two-tail	2 131450856	

Table 4.4.10 Comparative Results for MAD for Intrinsically Linear Model and NHPP Model

F-Test Two-Sample for Variances

	<i>NHPP Model</i>	<i>Intrinsically Linear Model</i>
Mean	28 81858882	1 711493177
Variance	7 220458921	0 006101305
Observations	13	13
df	12	12
F	1183 428565	
P(F<=f) one-tail	1.66731E-16	
F Critical one-tail	2 686633138	

t-Test Two-Sample Assuming Unequal Variances

	<i>NHPP Model</i>	<i>Intrinsically Linear Model</i>
Mean	28 81858882	1 711493177
Variance	7.220458921	0 006101305
Observations	13	13
Hypothesized Mean	0	
df	12	
t Stat	36.35706789	
P(T<=t) one-tail	6.00393E-14	
t Critical one-tail	1.782286745	
P(T<=t) two-tail	1.20079E-13	
t Critical two-tail	2 178812792	

Table 4.4.11 Predictive Validity Test Results for Proposed and Benchmark Models (Original and Simulated Data Sets)

Model	Inter group Test	Intra group Test	
		Price wise Test	Day wise Test
Intrinsically Linear Model	1.98	2.16	2.01
Swami and Khairnar(2003) Model	2.30	2.23	2.41
Tretheway and Weinberg (1991) Model	2.58	2.77	2.52
van Ryzin (1994) Model	2.34	2.48	2.38
NHPP Model	29.56	30.58	32.18
Neural Network Model	1.56	1.20	1.34

Note: Average MAD values are reported here

CHAPTER 5

Pricing Policies

5.1 Introduction

Price has operated as the major determinant of buyer choice. Price is one of the most flexible elements of marketing mix. At the same time, price competition is the number one problem facing companies. Yet companies do not handle pricing well. The most common mistakes are – price is not revised often enough to capitalize on market changes, and is not varied enough for different products items, market segments and purchase occasions. Many companies try to set a price that will maximize current, not long term, profits. They estimate the demand and costs associated with alternative prices and choose the price that produces maximum current profit, cash flow or rate of return on investments. This strategy assumes that the firm has knowledge of its demand and cost function (Kotler 2000).

It is a common practice for companies to price their products differently for different consumers. For example, airlines commonly charge different amounts for the same tickets (Belobaba 1987). The amount paid by customers may depend on when the ticket is purchased. Similarly, seasonal products like style goods are priced differently during different stages of the season. For perishable products such as airline tickets and fashion goods, different prices are widely used by the sellers. The price option is often rendered through the differentiation of the time when a good is purchased and by the amount of unsold inventory firms may have on hand. Price decisions on perishable products are affected by the length of time remaining before products are spoiled and by the levels of the unsold inventory.

In optimal pricing problem, it is the buyer who decides whether or not to buy the product at the list price. The seller sets the selling price at the beginning of the sales period based on the expected demand during the entire selling period. The seller may improve its revenue by varying the price from one period to another by considering the inventory position. The optimal pricing policy problem is closely related to the newsboy problem in inventory theory. In newsboy problem, the seller is to determine the optimal

supply level under the assumptions of the stochastic demand and the fixed product price. In optimal pricing problem major decision variable is list price.

5.2 Markovian Decision Process (MDP) Model

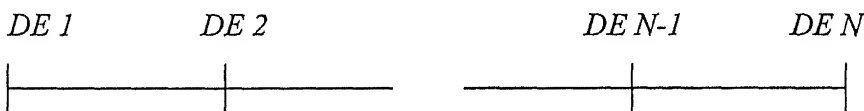
In MDP, a decision maker is faced with a problem of influencing the behavior of probabilistic systems as it evolves through time (Puterman 1994). He does this by making decisions or choosing actions. His goal is to choose a sequence of actions which causes the system to perform optimally with respect to some predetermined performance criterion. Since the system to be modeled is ongoing, the state of the system prior to tomorrow's decision depends on today's decision. Thus decisions must not be made myopically, but must anticipate the opportunities and rewards associated with future system states. MDP model consists of five elements: decision epochs, states, actions, transition probabilities and rewards. We propose to use MDP model for optimal pricing because demand in our case is stochastic in nature and pricing decision taken today affects revenue in future. MDP model elements are defined in conjunction with car rental business situation.

1. Decision Epoch

Decisions are made at points of time referred to as decision epochs (DE). Let T denote the set of decision epochs. In our case there are eight decision epochs i.e. number of days before reservation on which decision of pricing has to be taken.

$$T = \{0, 2, 4, 6, 8, 10, 12, 14\}$$

These decision epochs are discrete in nature. When discrete, decisions are made at all decision epochs. In discrete case, time is divided into periods or stages. We formulate models so that a decision epoch corresponds to the beginning of a period.



The set of decision epoch is finite. Thus we formulate our problem as finite horizon problem

2. State

At each decision epoch, the system occupies a state. We denote the set of possible system states by s in car rental data. If we assume the fleet size of 50 then system states are

$$S = \{0, 1, 2 \dots 50\}$$

3. Action

We observe the system in state s and choose an action a from the set of allowable actions in state s , A_s . In our case, action is choosing a price out of five possible prices.

Thus set of actions include choosing five different prices.

$$A = \{a_1, a_2, a_3, a_4, a_5\}$$

where a_1 corresponds to selection of first price and so on.

4. Rewards

The reward depends on the system state and action chosen. Let $r_t(s, a)$ defined for s and a denotes the reward at time t in state s for action a . In our case reward is the revenue generated. We do not consider any costs associated, then

$$r_t(s, a) = p * \min(s, E[D(p)])$$

where

p is the price chosen, s is number of cars remaining and $E[D(p)]$ represents expected demand for period t

5. Transition Probability

When decision maker takes an action a in state s , the system state at the next decision epoch is determined by the probability distribution $P_t(j | s, a)$. This function is called transition probability function. In simple words, transition probability is the probability that system will be in state j when action a is taken at decision epoch t and system state s

Let ,

s = number of cars available at beginning of period t .

J = number of cars available for $t+1$ period

d = demand for cars

$$P_t(J | s, a) = \begin{cases} 0 & \text{if } M \geq J \geq s - d \\ P_J & \text{if } M \geq J = s - d \end{cases}$$

First statement shows that probability of demand being nonnegative is zero.

Second statement shows probability that for a positive demand what will be the next stage.

The transition probability in stage t is equal to probability of demand in previous stage $t-1$ because $J = s - d$

The probability of demand can be calculated by using the NHPP arrival of customers

5.3 MDP Approach Example

We consider the optimal pricing problem for the car rental business.

We denote,

s_k – denotes state of the system and in our case it is number of cars available at the beginning of the k^{th} period.

a_k – denotes the action chosen at period k .

d_k – denotes demand during k^{th} period

Figure 5.3 1 shows the dynamics of the system considered. There are s_k cars available at period k . Demand for the cars is d_k in period k , so that number of cars remaining for the next period $k+1$ is s_{k+1} and is given by

$$s_{k+1} = s_k - d_k \quad (5.3.1)$$

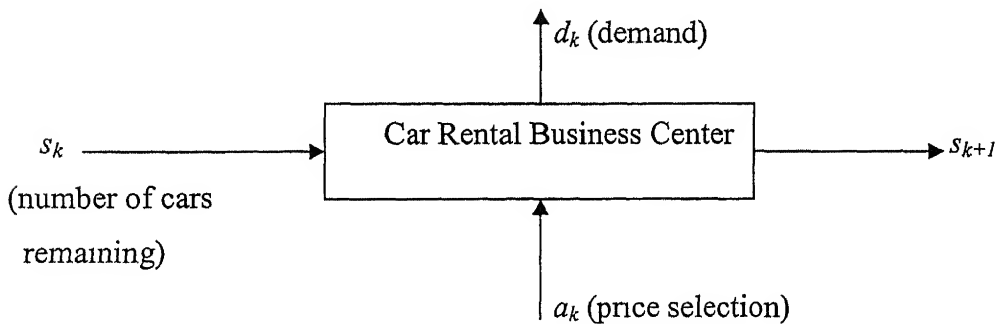


Figure (5.3.1) MDP Approach for Car Rental Pricing Policy

The revenue obtained at each period k consists of price selected and demand in that period. Total expected revenue over N period is given as,

$$\text{Expected Revenue} = E \left\{ \sum_{k=0}^{N-1} a_k * d_k \right\} \quad (5.3.2)$$

Our objective is to maximize revenue by proper choice of prices indicated by a_1 , a_2 , and a_3 . In other words, we are interested to determine an optimal policy rule for choosing a price at each period. Mathematically, the problem is one of finding a sequence of functions μ_k , $1k = 0, 1.. N-1$, mapping d_k into a_k to maximize the total expected revenue. The sequence $\mathfrak{R} = \{\mu_0, \mu_1, \dots, \mu_{N-1}\}$ is referred to as a policy. For each such \mathfrak{R} , the corresponding revenue for fixed initial number of cars s_0 is given by,

$$J_{\mathfrak{R}}(s_k) = E \left\{ \sum_{k=0}^{N-1} a_k * d_k \right\} \quad (5.3.3)$$

We assume there are five cars available at the beginning of the period, car rental business offers three prices, namely, \$19, \$24, and \$29, and the probability distribution function for the demand is known to the seller a priori. The figure 5.3.2 shows relationship among state, action and rewards for our example

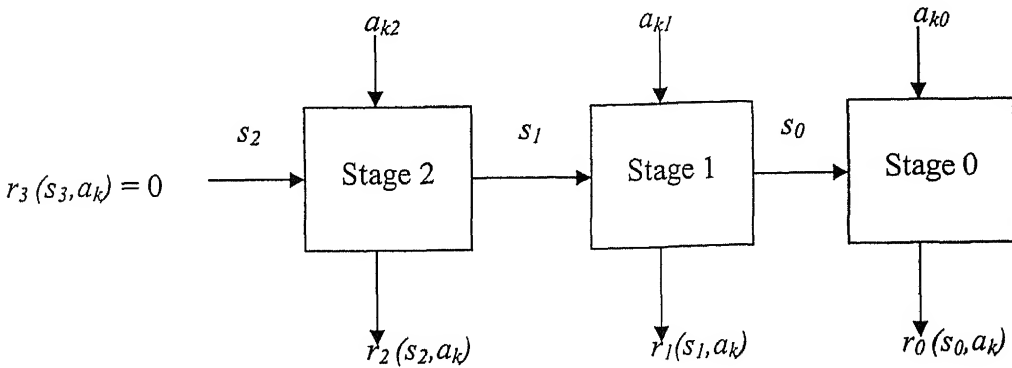


Figure (5.3.2): Relationship among State, Action, and Reward for Dynamic System

The optimal pricing policy is of the form

$$J_k(d_k) = \max E \{ (a_k * d_k) + J_{k+1}[\max(0, d_k)] \}, k = 0, 1, 2, \dots \quad (5.3.4)$$

We now present the optimal policy in Table 5.3.1. Appendix E contains detail calculations of the problem

Table 5.3.1 Optimal Pricing Policy for Car Rental Business

Number of cars remaining	Stage 0		Stage 1		Stage 2	
	Revenue	Optimal Price	Revenue	Optimal Price	Revenue	Optimal Price
1	45.28	\$29	32.76	\$29	17.4	\$29
2	68.11	\$24	45.9	\$29	22.8	\$29
3	118.29	\$24	80.09	\$24	40.89	\$29
4	158.56	\$19	112.78	\$24	54	\$24
5	208.54	\$19	139.82	\$19	73.95	\$24

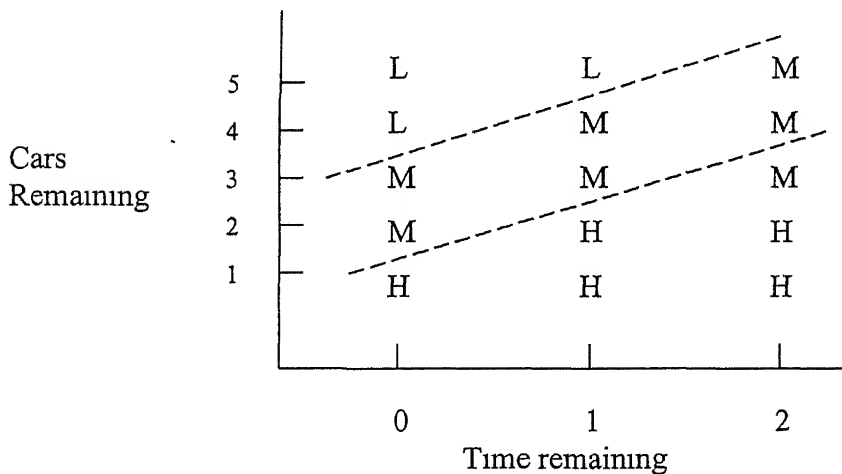


Figure (5.3.3): Optimal Pricing Policy

Effect of cars remaining and time remaining on optimal pricing is depicted in Figure 5.3.3. The letters L, M, and H represent low, medium, and high prices respectively. Optimal pricing policies based on time remaining and cars remaining is shown by the dotted lines. At the beginning of the period, though cars remaining are more, medium prices can be offered. Lower prices are not offered in the beginning

This is justifiable because at the beginning of the period, car rental business can take calculated risks by not allowing lower prices. When cars remaining are less, higher price option is implemented. It is observed in the above example that, when number of cars remaining is close to one, highest price option leads to more revenue generation, irrespective of time remaining. Lower price option is offered when time remaining is less and number of cars remaining is more. In practice, when products are not sold till the deadline, then selling products at discounts is good option.

CHAPTER 6

Managerial Implications

The hierarchy of managerial problems in yield management is presented in the following Figure 6 1

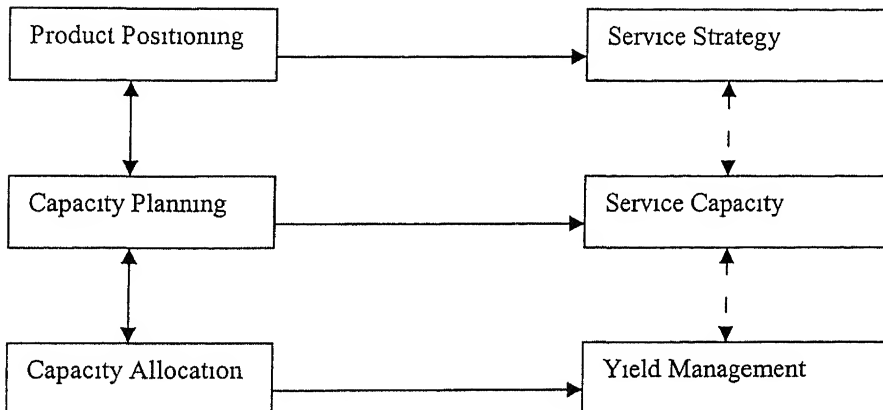


Figure (6.1). Hierarchy of Managerial Problems in Yield Management

(Source Sergi S. 2001)

Yield management forms the foundation of the managerial problems. The objectives for a yield manager are profit maximization, maximize capacity utilization and maximize the revenue from sale of perishable product or services. One important function of the yield manager is to extract the maximum price from each customer without losing the goodwill. The constraints faced by the yield manager in achieving these objectives are capacity constraints, marketing constraints and strategic constraints.

To achieve these objectives, the yield manager has to make certain decisions. These decisions help to overcome the conflicting objectives like maximize capacity utilization and simultaneously increase the revenue. If the yield manager tries to enhance the capacity utilization by offering lower prices, then the objective of the maximization of the revenue is hampered. On the other hand, if he sets the prices higher, he may be able to generate higher revenue but one cannot guarantee effective capacity utilization. Thus, the yield manager is forced to make certain decisions like demand estimation, pricing, determining the capacity, determining when to offer discounts and how much to offer as discount for promoting the product or service.

This thesis fits well in the context of decision making in yield management. We specifically deal with two important decisions viz demand estimation and pricing policies.

First part of the thesis deals with demand estimation. Yield manager's objective is to sell the product or service to the customers having high valuation so that high margins are achieved. At the same time he cannot wait too long for arrival of the high valuation customers. Thus to make a trade off, demand estimation is necessary. We have proposed different demand estimation models. In the demand estimation models the information needed for the demand estimation is price offered and corresponding demand on the given day. Using this information yield manager can predict the demand well in advance. Intrinsically linear model is found best from the predictive validity tests for demand estimation. We have also proposed the neural network approach for demand estimation. For this approach to use, it is imperative for the yield manager to update the data regularly for the neural network training purposes. Every time new data is entered, data training is necessary. Thus different models proposed can be used to estimate the demand which is important input for the pricing decisions.

Second part of the thesis deals with the pricing policies. Price is one of the most flexible elements of the marketing mix and operates as the major determinant of buyer choice. Pricing decisions in yield management are governed by the time of sale and the remaining inventory of the product. We have proposed Markovian Decision Process approach to determine optimal pricing. This approach is more realistic because demand is stochastic in nature. In this approach, we have proposed pricing in a dynamically changing system in which decisions are not myopic. This feature of the approach is important from the management point of view. Demand is stochastic in nature and today's pricing decision affects tomorrow's decision. The MDP approach is more realistic for optimal pricing. We have proposed the optimal pricing which helps yield manager to take decision of price change in accordance with inventory remaining and time remaining. In the beginning of the selling period, lower prices are not offered. Medium and high prices are offered in the beginning. As the demand evolves over time and if product inventory is low, then higher prices are offered to generate higher revenues.

When inventory of the products is more, lower prices are offered towards the end of selling period. Discounts may be offered to accelerate the demand

Yield management can give a firm a competitive edge, but it could also result in a loss of competitive focus. Since most yield management systems focus on maximizing revenue, companies using such a system may develop undue focus on short term profits and ignore long term profits which could result from managerial attention to producing and delivering good services (Kimes S.1991). Many service organizations are successful because they offer very high quality services which are in high demand. The focus on efficient resource use that yield management implies may take managerial attention away from customer service and fundamentally change the service concept.

Customer alienation results in yield management practice because consumers seem to be resigned to the fact that airline charge different prices depending on how far ahead a ticket was bought, and on what restrictions were met. In industries with only a few major competitors, like airline and car rental services, this may work, but in industries with large competitors such as the hotel industry, a customer who does not like paying different prices for the same room may decide to support the competitor. Similarly, customers may find it unfair to be paying a higher price for a service than someone who reserved it a few time earlier.

Yield management systems make decisions about how many items to sell at what price. This job also involves expertise and experience of the yield manager. Unless properly structured to allow for some judgment, yield management systems could be met with resentment from people using it. Sales people are generally rewarded by the amount of sale. Yield management could also cause a problem for sales department. In yield management it is not only the number of products sold counts, but at what price they are sold also counts. Thus rewarding sales people just looking at sales volume dose not solve the purpose. Mangers are often rewarded on the basis of capacity utilization or average rate. With yield management, the manger needs to be concerned with both these factors. Unless the performance intensive system is changed to reflect this, managers may resent using yield management. Yield management requires 'extensive training of all the employees. The employees must clearly understand the purpose of the yield management and essentially how it works. They must be made clear about how yield management is

going to change their jobs. Top management can not expect that yield management will just happen, but it requires careful planning and training of the employees.

One of the major problems confronting the firms by application of yield management would be the degree of centralization of reservation systems. Airlines have traditionally had highly centralized reservation systems, but other industries such as hotel, car rental agencies and freight shipment industries may face problems for centralized reservation systems. Lastly without the top management support and commitment yield management would not give the desired results, making top management commitment an essential part of yield management.

CHAPTER 7

Conclusions, Limitations and Directions for Future Research

7.1 Conclusions

In this thesis we proposed demand estimation models in the first part and optimal pricing policies in the second part. For demand estimation we have proposed four models and NHPP and neural network approach for demand. Similarly, for pricing policies we have proposed MDP approach. We have decomposed the demand into three factors as price elasticity, time of sale and influence of remaining inventory. In literature of economics, demand is assumed as function of price (van Ryzin, 1993). Previous demand models are proposed considering price elasticity only. In this thesis, we proposed to add the effect of time of sale and influence of remaining inventory. We tried to capture the effect of time of sale on demand. Our results show that the time of sale plays an important role in demand.

We add a new dimension in demand estimation by introducing the effect of inventory remaining. We tried to show that the scarcity effect is present in the perishable products also. In particular, as the number of products remaining decreases, the perceived value of the product increases and it results in higher demand. We have shown this effect by considering the term inventory remaining. The results show that previous sales are significant in estimating demand. We have proposed intrinsically linear model with this approach. In this model we assume that the seller has sufficient information regarding all the relevant factors. Predictive validity tests show that intrinsically linear model is more robust for variation in demand patterns. We have also proposed and analyzed Swami and Khairnar (2003) model, Tretheway and Weinberg (1991) model, and van Ryzin (1994) model.

In neural network approach, we have used back-propagation algorithm. This algorithm is able to represent any functions (Law 1999). Back-propagation algorithms are more flaw tolerant than regression models. Results of neural network approach are impressive. However, neural networks do not structure knowledge with mathematical functions. The weights of neural networks may be interpreted as meaningless numbers which do not serve any purpose for analysis. This is not the case with regression analysis, where coefficients of model provide useful information regarding mathematical

Our model assumes that demand is independent of such effects. In reality, substitution effects may be present. In simple words we are assuming no segmentation of the market.

7.3 Directions for Future Research

Several areas of research emerge from the present study. One direction for research is the introduction of the competitive effects present in the market. Competition affects the pricing decisions also. To introduce this effect one needs demand data for various firms competing in the same market at the same time. Substitution effects can also be introduced with this data. Quality of service is also an important factor governing the demand process. If quality is known in a predictable manner, then it can be incorporated in the model. The brand name also plays an important role in demand. Promotions and sales efforts help in building the brand name of the product. It helps in generating awareness and this may increase the probability of adoption of the product. Introduction of these factors can lead to more realistic demand estimation models.

In the present study we have assumed three factors which govern demand. Demand estimation results would be more significant if advertising effect was also considered. This feature would be of great help to the yield manager. The yield manager would be in a better position for decisions regarding levels of advertising required for the promotion of the product or service (Tretheway and Weinberg 1991). For optimal pricing policies, future research can be done by considering the substitution effects in the

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Appendix A

Simulated Data Sets

A1. Data Set 1 – Low Variation:

	Days before reservation	0	2	4	6	8	10	12	14
	Price								
Replicate 1	19	43	36	38	23	28	32	18	9
	24	44	40	32	26	21	21	22	16
	29	39	42	37	32	30	15	5	3
	34	34	29	24	27	18	16	9	5
	39	19	21	17	17	10	3	2	1

Replicate 2	19	46	45	38	26	23	30	15	8
	24	38	44	42	20	23	27	16	6
	29	46	34	30	31	34	11	18	6
	34	43	32	28	28	16	22	15	9
	39	22	24	21	16	8	10	0	0

Replicate 3	19	43	48	32	30	27	23	17	9
	24	49	42	35	35	26	23	18	8
	29	39	38	34	21	25	18	14	6
	34	33	38	26	20	7	17	17	9
	39	35	22	11	16	9	0	0	1

Replicate 4	19	50	46	40	40	27	23	11	12
	24	32	34	40	36	28	18	16	5
	29	34	25	26	24	23	21	17	7
	34	30	34	33	16	22	12	8	10
	39	35	16	26	20	20	11	3	7

A2. Data Set 1 – Medium Variation:

	Days before reservation	0	2	4	6	8	10	12	14
	Price								
Replicate 1	19	54	40	44	20	29	33	15	6
	24	46	27	43	21	32	32	20	6
	29	40	37	36	9	32	2	26	16
	34	51	7	20	15	14	20	-4	2
	39	27	7	17	9	-2	7	20	8

Replicate 2	19	60	68	40	44	20	32	24	16
	24	64	23	39	35	35	18	19	9
	29	46	40	27	49	37	4	24	12
	34	31	18	15	19	20	17	9	6
	39	38	15	13	3	3	-2	16	12

Replicate 3	19	53	45	35	32	37	36	28	19
	24	37	39	35	28	37	17	21	14
	29	47	48	43	26	31	18	4	2
	34	44	26	25	44	24	19	17	9
	39	28	28	35	19	12	26	-3	0

Replicate 4	19	63	15	36	15	14	17	13	6
	24	54	57	39	27	32	18	4	15
	29	30	40	32	39	21	27	17	2
	34	23	24	40	24	41	9	-1	0
	39	35	31	19	14	7	22	2	7

A3. Data Set 1 – High Variation:

	Days before reservation	0	2	4	6	8	10	12	14
	Price								
Replicate 1	19	42	38	83	37	5	31	-1	0
	24	38	40	17	25	4	31	-23	-12
	29	29	72	6	73	35	21	-2	-2
	34	68	12	5	33	20	28	-1	0
	39	15	5	28	15	-21	11	16	8

Replicate 2	19	72	49	48	8	27	22	21	19
	24	58	55	53	18	7	16	25	14
	29	28	17	25	27	22	47	-34	0
	34	25	45	18	34	37	21	-10	1
	39	36	32	27	18	1	26	1	1

Replicate 3	19	38	39	67	26	59	25	6	9
	24	33	45	50	69	45	30	2	26
	29	42	38	83	37	5	31	-1	0
	34	38	40	17	25	4	31	-23	-12
	39	29	72	6	73	35	21	-2	-2

Replicate 4	19	72	49	48	8	27	22	21	19
	24	43	37	17	23	51	31	3	15
	29	68	12	5	33	20	28	-1	0
	34	15	5	28	15	-21	11	16	8
	39	23	41	29	32	10	40	28	-4

A4. Data Set 2 – Low Variation:

	Days before reservation	0	2	- 4	6	8	10	12	14
	Price								
Replicate 1	19	48	52	44	36	19	14	3	11
	24	47	40	38	41	23	19	25	12
	29	43	34	31	30	21	23	22	11
	34	38	30	24	17	26	21	21	11
	39	28	21	22	14	3	26	9	11

Replicate 2	19	50	37	50	30	18	15	11	12
	24	43	46	39	30	29	18	16	20
	29	40	41	42	23	21	29	13	10
	34	33	32	29	17	14	21	20	5
	39	21	20	15	15	11	14	13	10

Replicate 3	19	47	47	38	33	23	15	14	19
	24	39	36	39	35	28	23	25	13
	29	47	34	32	33	23	28	22	13
	34	44	22	27	22	27	23	19	1
	39	19	12	19	15	14	27	15	8

Replicate 4	19	44	37	43	40	23	26	17	11
	24	38	34	39	34	25	32	18	10
	29	45	32	27	30	22	21	8	14
	34	34	28	33	22	22	14	11	-5
	39	29	21	25	14	6	6	-3	-2

A5. Data Set 2 – Medium Variation:

	Days before reservation	0	2	4	6	8	10	12	14
	Price								
Replicate 1	19	37	26	38	18	25	36	19	12
	24	44	37	25	18	18	18	27	12
	29	38	51	44	38	36	15	-2	0
	34	31	28	16	27	14	20	6	1
	39	5	18	13	18	9	-6	-5	2

Replicate 2	19	43	44	38	24	15	33	12	10
	24	33	44	45	6	22	30	15	2
	29	52	35	31	36	45	7	23	12
	34	49	34	26	31	10	31	19	21
	39	12	22	20	16	4	9	-18	0

Replicate 3	19	63	53	37	16	29	12	36	25
	24	40	35	29	16	44	10	9	8
	29	46	26	21	32	22	16	-1	0
	34	34	10	5	27	24	16	13	16
	39	10	29	12	6	-4	-9	5	2

Replicate 4	19	38	61	69	33	22	19	13	28
	24	33	30	43	34	12	23	20	12
	29	46	39	38	26	16	22	0	2
	34	25	28	29	26	21	20	16	24
	39	23	38	4	18	13	10	-5	-1

A6. Data Set 2 – High Variation:

	Days before reservation	0	2	4	6	8	10	12	14
	Price								
Replicate 1	19	26	58	8	27	21	15	16	8
	24	65	40	25	37	32	25	22	12
	29	37	52	46	3	32	20	20	9
	34	24	60	15	10	-30	30	37	17
	39	53	16	-14	21	4	-27	-20	0

Replicate 2	19	57	35	49	13	28	37	12	4
	24	49	10	47	8	42	39	23	6
	29	42	42	41	-7	42	-11	40	15
	34	65	-16	10	5	8	27	-19	-14
	39	21	-11	13	2	-14	2	31	18

Replicate 3	19	70	90	40	61	9	35	31	15
	24	85	3	40	34	47	11	19	12
	29	52	47	24	71	52	-8	36	23
	34	25	6	-2	11	18	21	7	8
	39	43	5	5	-11	-6	-16	23	2

Replicate 4	19	33	37	39	74	7	4	62	-9
	24	59	43	18	-7	13	9	26	34
	29	41	46	34	27	3	18	30	1
	34	33	25	27	21	30	19	15	4
	39	-6	5	15	19	41	18	-1	0

A7. Data Set 3 – Low Variation:

	Days before reservation	0	2	4	6	8	10	12	-14
	Price								
Replicate 1	19	51	46	38	37	35	32	26	12
	24	42	39	36	31	32	21	18	14
	29	46	44	38	28	31	18	7	6
	34	37	26	21	29	16	14	13	12
	39	30	26	28	19	12	17	0	1

Replicate 2	19	51	50	40	38	33	28	20	18
	24	45	38	45	43	27	28	12	15
	29	43	39	36	28	31	15	20	23
	34	30	30	21	10	12	15	23	14
	39	36	21	22	9	4	6	3	1

Replicate 3	19	47	44	51	38	24	27	16	10
	24	43	41	32	32	21	26	6	2
	29	39	46	25	40	29	18	8	5
	34	41	24	17	20	14	15	6	8
	39	27	19	22	16	3	9	6	6

Replicate 4	19	47	43	33	43	36	28	20	26
	24	47	41	38	27	30	22	12	17
	29	44	42	33	31	24	18	13	20
	34	30	25	25	23	14	19	0	6
	39	32	29	28	12	7	7	2	-12

A8. Data Set 3 – Medium Variation:

	Days before reservation	0	2	4	6	8	10	12	14
	Price								
Replicate 1	19	61	48	44	25	30	24	22	12
	24	52	47	45	26	17	21	20	14
	29	36	28	29	28	26	32	-12	10
	34	27	35	18	24	23	15	-1	1
	39	34	29	24	18	7	17	2	0

Replicate 2	19	53	63	47	36	19	32	6	3
	24	51	40	40	51	18	28	8	2
	29	46	35	34	32	25	23	10	8
	34	40	29	21	13	31	3	13	1
	39	37	24	31	21	-2	5	20	15

Replicate 3	19	56	33	59	24	17	13	7	5
	24	43	53	40	29	31	9	24	12
	29	41	49	55	18	25	20	7	17
	34	29	33	31	12	7	12	0	0
	39	23	22	17	21	15	9	19	10

Replicate 4	19	52	70	48	34	17	21	11	11
	24	47	32	49	52	15	30	9	8
	29	52	44	34	29	30	-4	12	5
	34	21	35	24	25	14	11	31	8
	39	12	11	8	9	17	7	0	14

A9. Data Set 3 – High Variation:

	Days before reservation	0	2	4	6	8	10	12	14
	Price								
Replicate 1	19	54	58	30	24	22	39	22	15
	24	27	26	45	43	32	30	5	4
	29	67	33	41	48	33	19	14	10
	34	69	-6	32	27	53	19	-25	0
	39	8	-11	27	26	31	-34	23	15

Replicate 2	19	31	34	72	40	34	22	43	37
	24	53	27	0	49	11	14	47	45
	29	41	46	29	27	5	-10	-3	1
	34	41	47	-1	18	39	53	-8	-8
	39	87	25	34	59	25	14	1	0

Replicate 3	19	41	32	50	44	8	14	25	12
	24	-13	52	47	21	31	23	0	1
	29	7	35	37	36	41	-7	17	18
	34	8	20	38	-10	47	31	7	6
	39	7	46	-20	6	-13	-10	10	18

Replicate 4	19	60	20	50	36	16	34	-5	-12
	24	11	6	27	42	36	33	23	-3
	29	45	3	50	10	34	16	19	67
	34	18	7	48	32	10	1	-18	48
	39	40	34	26	31	-29	15	-9	11

A10. Data Set 4 – Low Variation:

	Days before reservation	0	2	4	6	8	10	12	14
	Price								
Replicate 1	19	49	41	42	47	26	31	25	12
	24	36	36	34	31	30	19	11	8
	29	53	35	34	23	26	12	13	15
	34	30	25	21	22	13	16	7	8
	39	23	27	23	13	11	7	4	1

Replicate 2	19	43	46	39	35	35	21	20	16
	24	36	41	35	36	27	19	16	14
	29	37	33	26	31	19	11	19	18
	34	27	35	26	24	10	1	5	1
	39	29	20	22	9	9	8	5	0

Replicate 3	19	50	47	48	33	31	24	17	18
	24	49	47	36	32	28	14	20	16
	29	34	38	25	38	22	15	10	12
	34	39	39	20	22	9	3	3	1
	39	22	17	7	14	11	4	2	2

Replicate 4	19	42	44	37	41	36	32	14	15
	24	41	36	42	31	27	26	11	16
	29	34	36	29	23	11	7	8	16
	34	34	36	23	17	13	5	9	12
	39	25	22	10	-1	12	11	5	-3

A 11. Data Set 4 – Medium Variation:

	Days before reservation	0	2	4	6	8	10	12	14
	Price								
Replicate 1	19	29	69	50	42	34	24	23	12
	24	32	35	26	43	52	20	16	14
	29	38	42	22	25	5	20	12	10
	34	35	15	21	4	9	17	19	18
	39	26	36	5	-1	2	3	9	3

Replicate 2	19	62	33	35	31	16	36	9	6
	24	35	32	34	28	28	24	9	7
	29	30	47	19	17	22	14	17	10
	34	33	18	46	19	2	19	-7	-1
	39	16	21	23	20	12	8	9	6

Replicate 3	19	50	60	37	41	50	25	8	1
	24	49	40	31	39	36	22	9	7
	29	55	34	36	24	15	1	23	12
	34	27	23	16	4	11	10	24	16
	39	30	18	24	11	-11	26	3	2

Replicate 4	19	49	35	53	31	34	16	23	17
	24	55	39	31	38	29	10	18	22
	29	37	33	27	13	34	7	6	-10
	34	20	26	16	16	-1	21	23	-5
	39	20	44	9	8	19	16	-8	20

A12. Data Set 4 – High Variation:

	Days before reservation	0	2	4	6	8	10	12	14
	Price								
Replicate 1	19	60	39	3	30	38	59	-2	-1
	24	48	23	55	15	43	24	14	12
	29	56	52	32	29	7	37	18	10
	34	29	7	36	-8	16	8	9	5
	39	23	11	15	63	2	3	30	12

Replicate 2	19	63	43	33	29	48	24	-1	0
	24	50	24	28	18	42	29	-5	-2
	29	41	56	12	19	41	15	12	6
	34	30	42	-2	23	11	-6	30	14
	39	47	24	21	19	2	6	-19	-10

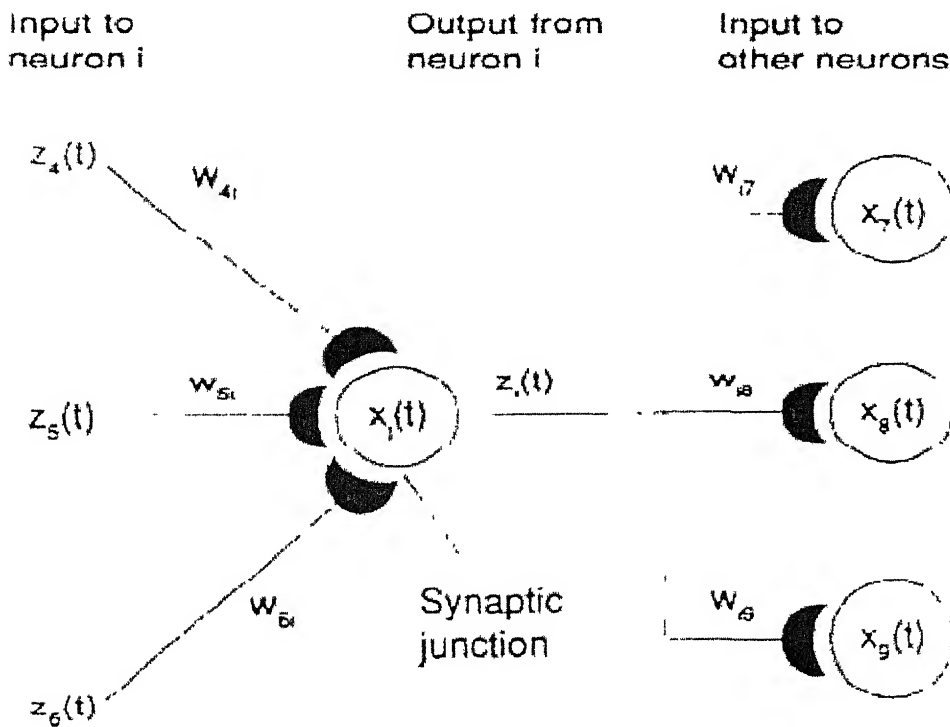
Replicate 3	19	51	76	26	75	50	56	25	12
	24	47	30	32	36	13	19	2	1
	29	37	42	48	8	1	6	25	12
	34	56	-6	35	7	24	-20	-15	1
	39	11	41	16	16	55	3	-19	0

Replicate 4	19	55	48	20	96	15	17	-2	4
	24	54	53	65	20	13	45	17	30
	29	9	56	45	37	20	-2	40	3
	34	33	37	25	27	13	4	26	8
	39	35	7	14	36	29	-37	37	-7

Appendix B

The back-propagation Algorithm - a Mathematical Approach

Units are connected to one another. Connections correspond to the edges of the underlying directed graph. There is a real number associated with each connection, which is called the weight of the connection. We denote by W_{ij} the weight of the connection from unit u_i to unit u_j . It is then convenient to represent the pattern of connectivity in the network by a weight matrix W whose elements are the weights W_{ij} . Two types of connection are usually distinguished: excitatory and inhibitory. A positive weight represents an excitatory connection whereas a negative weight represents an inhibitory connection. The pattern of connectivity characterizes the architecture of the network.



A unit in the output layer determines its activity by following a two step procedure

- 1 First, it computes the total weighted input x_j , using the formula

$$X_j = \sum_i y_i W_{ij}$$

where y_i is the activity level of the i th unit in the previous layer and W_{ij} is the weight of the connection between the i th and the j th unit

- 2 Next, the unit calculates the activity y_j using some function of the total weighted input. Typically we use the sigmoid function

$$y_j = \frac{1}{1 + e^{-x_j}}$$

Once the activities of all output units have been determined, the network computes the error E , which is defined by the expression

$$E = \frac{1}{2} \sum_j (y_j - d_j)^2$$

where y_j is the activity level of the j th unit in the top layer and d_j is the desired output of the j th unit

The back-propagation algorithm consists of four steps

- 1 Compute how fast the error changes as the activity of an output unit is changed. This error derivative (EA) is the difference between the actual and the desired activity

$$EA_j = \frac{\partial E}{\partial y_j} = y_j - d_j$$

- 2 Compute how fast the error changes as the total input received by an output unit is changed. This quantity (EI) is the answer from step 1 multiplied by the rate at which the output of a unit changes as its total input is changed

$$EI_j = \frac{\partial \mathcal{E}}{\partial x_j} = \frac{\partial \mathcal{E}}{\partial y_j} \times \frac{dy_j}{dx_j} = EA_j y_j (1 - y_j)$$

3 Compute how fast the error changes as a weight on the connection into an output unit is changed. This quantity (EW) is the answer from step 2 multiplied by the activity level of the unit from which the connection emanates

$$EW_v = \frac{\partial \mathcal{E}}{\partial W_v} = \frac{\partial \mathcal{E}}{\partial x_j} \times \frac{x_j}{W_v} = EI_j y_i$$

4. Compute how fast the error changes as the activity of a unit in the previous layer is changed. This crucial step allows back propagation to be applied to multi layer networks. When the activity of a unit in the previous layer changes, it affects the activities of all the output units to which it is connected. So to compute the overall effect on the error, we add together all these separate effects on output units. But each effect is simple to calculate. It is the answer in step 2 multiplied by the weight on the connection to that output unit

$$EA_i = \frac{\partial \mathcal{E}}{\partial x_i} = \sum_j \frac{\partial \mathcal{E}}{\partial x_j} \times \frac{x_j}{x_i} = \sum_j EI_j W_{vj}$$

By using steps 2 and 4, we can convert the EAs of one layer of units into EAs for the previous layer. This procedure can be repeated to get the EAs for as many previous layers as desired. Once we know the EA of a unit, we can use steps 2 and 3 to compute the EWs on its incoming connections

APPENDIX C

'C' Language Code for NHPP Model

```
/* Computer Code for NHPP Process */
#include <stdio h>
#include <stdlib h>
#include <math h>
#define MAXVAL 110
main()
{
    int t,i,n,rows=0,j,
    int factorial=0,
    int jpos=0;
    float p1,p2,p3,p4,p5,p6,
    int days_reserve,price_level,
    int a=0,b=0;
    int count=0;
    float temp=0,c=0,
    float max_prob=0,
    float sum_prob=0;
    double c1,n1,factorial1,

    double prob[MAXVAL],

    /* enter parameters found for lamda */
    printf("Enter parameter1."),
    scanf("%f",&p1);

    printf("Enter parameter2 "),
    scanf("%f",&p2);

    printf("Enter parameter3 ");
    scanf("%f",&p3),

    printf("Enter parameter4 "),
    scanf("%f",&p4),

    printf("Enter parameter5 "),
    scanf("%f",&p5);

    printf("Enter parameter6."),
    scanf("%f",&p6);
    /* enter any price level 1 for 19, 2 for 24 */
    printf("Enter Price Level."),
    scanf("%d",&price_level),

    printf("Enter Days Before Reservation(among 0,2,4,6,8,10,12) "),
    scanf("%d",&days_reserve),

    printf("Enter maximum number of customers."),
    scanf("%d",&n);

    if(price_level>=6)
```

```
        {printf("\nERROR IN THE PRICE LEVEL, KINDLY CHOOSELEVEL BETWEEN 1  
TO 5\n");
```

```
    if(price_level==1)  
    { p3=0;  
      p4=0;  
      p5=0,  
      p6=0,  
    }
```

```
    if(price_level==2)  
    {  
      p4=0,  
      p5=0,  
      p6=0;  
    }
```

```
    if(price_level==3)  
    { p3=0,  
      p5=0;  
      p6=0;  
    }
```

```
    if(price_level==4)  
    { p3=0;  
      p4=0;  
      p6=0;  
    }
```

```
    if(price_level==5)  
    { p3=0;  
      p4=0;  
      p5=0;  
    }
```

```
    if(days_reserve==0)  
    { b=14;  
      a=12;  
    }
```

```
    if(days_reserve==2)  
    { b=12;  
      a=10,  
    }
```

```
    if(days_reserve==4)  
    { b=10,  
      a=8,  
    }
```

```
    if(days_reserve==6)  
    { b=8;  
      a=6;  
    }
```

```

    if(days_reserve==8)
    { b=6,
      a=4;
    }

    if(days_reserve==10)
    { b=4;
      a=2,
    }

    if(days_reserve==12)
    { b=2;
      a=0;
    }

/*operations*/

    c=0,
    c = ( b-a ) * ( p1 + 0.5*p2*(b*b-a*a) + p3 + p4 + p5 + p6),
    for(j=1,j<=n,j++)
    {prob[j]=0;
    }
    factorial=0;
    for(j=1,j<=n;j++)
    {factorial=factorial*j,
    }
    c1=(double)c,
    n1=(double)n;
    factorial1=(double)factorial,
    for(j=1;j<=n;j++)
    { prob[j]=pow(c1, n1) * (1/exp(c1))/factorial1;
    }

    max_prob=0;
    jpos=0;
    max_prob=prob[1];
    jpos=1;
    for(j=2;j<=n;j++)
    { if(max_prob<=prob[j])
      max_prob=prob[j];
      jpos=j,
    }

    for(i=1,i<=n,i++)
    { sum_prob=sum_prob+prob[i],
      if(sum_prob>0 1)
      {count=i,
       i=n;
      }
    }

    printf("\With 90percent confidence,demand will exceed %d\n",n,n),

}
}

```

APPENDIX D

MATLAB Code for Neural Network Training

```
Neural network program for training data sets.

x=load('ip.txt')
y=load('op.txt')

a=x ',
b=y ',

net = newff([0 50,19 34,0 12],[5 1],{'tansig' 'purelin'})

Y = sim(net,a);

plot(a,b,a,Y,'o')

net.trainParam.epochs = 1000;

net.trainParam.goal = 1,

net = train(net,a,b);

Y = sim(net,a);

plot(a,b,a,Y,'o')
```

APPENDIX E

MDP Approach Example

The optimal pricing policy is of the form

$$J_k(d_k) = \max E\{(a_k * d_k) + J_{k+1}[\max(0, d_k)]\}, k = 0, 1, 2, \dots$$

Transition probability matrix is presented in Table B 4 to Table B.6

Stage 2

For stage 2, $J_3[\max(0, d_k)] = 0$.

We compute $J(d)$ —revenue in stage 2 and for all possible states.

$$J_3[\max(0, d_k)] = 0$$

For $s_2 = 1$

$$J_2(1) = \max E\{(a_k * d_2) + J_3[\max(0, d_k)]\}$$

We calculate the expectation of the right side for each of the three prices

$$a_1 = \$19 \quad E\{.\} = 19 * 1 * 0.5 + 0 = 9.5$$

$$a_2 = \$24 \quad E\{.\} = 24 * 1 * 0.4 + 0 = 9.6$$

$$a_3 = \$29 \quad E\{.\} = 29 * 1 * 0.6 + 0 = 17.4$$

$$\text{Hence } J_2(1) = \$17.4; \quad \mu_2^*(1) = \$29$$

For $s_2 = 2$

$$J_2(2) = \max E\{(a_k * d_2) + J_3[\max(0, d_k)]\}$$

We calculate the expectation of the right side for each of the three prices.

$$a_1 = \$19 \quad E\{.\} = 19 * 2 * 0.2 + 19 * 1 * 0.4 = 22.8$$

$$a_2 = \$24 \quad E\{.\} = 24 * 2 * 0.2 + 24 * 1 * 0.4 = 21.6$$

$$a_3 = \$29 \quad E\{.\} = 29 * 2 * 0.3 + 29 * 1 * 0.5 = 31.9$$

$$\text{Hence } J_2(2) = \$31.9; \quad \mu_2^*(2) = \$29$$

For $s_2 = 3$

$$J_2(3) = \max E\{(a_k * d_2) + J_3[\max(0, d_k)]\}$$

We calculate the expectation of the right side for each of the three prices

$$a_1 = \$19 \quad E\{.\} = 19\{3 * 0.2 + 2 * 0.4 + 0.4\} = 28.5$$

$$a_2 = \$24 \quad E\{.\} = 24\{3 * 0.1 + 2 * 0.4 + 0.4\} = 36$$

$$a_3 = \$29 \quad E\{.\} = 29\{3*0.6 + 2*1 + 0.5\} = 20.3$$

$$\text{Hence } J_2(3) = \$17.4, \quad \mu_2^*(3) = \$29$$

For $s_2 = 4$

$$J_2(4) = \max E\{(a_k * d_2) + J_3[\max(0, d_k)]\}$$

We calculate the expectation of the right side for each of the three prices

$$a_1 = \$19 \quad E\{.\} = 19\{4*0.2 + 3*0.4 + 2*0.5 + 0.1\} = 39.9$$

$$a_2 = \$24 \quad E\{.\} = 24\{4*0.1 + 3*0.2 + 2*0.6 + 0.1\} = 54.0$$

$$a_3 = \$29 \quad E\{.\} = 29\{4*0.1 + 3*0.1 + 2*0.5 + 0.15\} = 53.65$$

$$\text{Hence } J_2(4) = \$54, \quad \mu_2^*(4) = \$24$$

For $s_2 = 5$

$$J_2(5) = \max E\{(a_k * d_2) + J_3[\max(0, d_k)]\}$$

We calculate the expectation of the right side for each of the three prices

$$a_1 = \$19 \quad E\{.\} = 19\{5*0.1 + 4*0.1 + 3*0.3 + 2*0.4 + 0.05\} = 50.35$$

$$a_2 = \$24 \quad E\{.\} = 24\{5*0.07 + 4*0.1 + 3*0.4 + 2*0.3 + 0.1\} = 73.95$$

$$a_3 = \$29 \quad E\{.\} = 29\{5*0.06 + 4*0.2 + 3*0.3 + 2*0.2 + 0.15\} = 66.48$$

$$\text{Hence } J_2(5) = \$73.95, \quad \mu_2^*(5) = \$24$$

Similar computations are done for Stage 1 and Stage 0. The computational results are tabulated below.

Table B.1 Stage 2 Revenue Computations

	Price \$19	Price \$24	Price \$29	Optimal Revenue	Optimal Price
$J_2(1)$	9.5	9.6	17.4	17.4	\$29
$J_2(2)$	22.8	21.6	31.9	31.9	\$29
$J_2(3)$	28.5	36	40.89	40.89	\$29
$J_2(4)$	39.9	54	53.65	54	\$24
$J_2(5)$	50.35	73.95	66.48	73.95	\$24

Table B.2 Stage 1 Revenue Computations

	Price \$19	Price \$24	Price \$29	Optimal Revenue	Optimal Price
$J_2(1)$	27.82	26.52	32.76	32.76	\$29
$J_2(2)$	41.43	42.3	45.9	45.9	\$29
$J_2(3)$	71.24	80.09	76.57	80.09	\$24
$J_2(4)$	94.28	112.78	102.26	112.78	\$24
$J_2(5)$	139.82	134.66	135.18	139.82	\$19

Table B.3 Stage 0 Revenue Computations

	Price \$19	Price \$24	Price \$29	Optimal Revenue	Optimal Price
$J_2(1)$	42.88	44.56	45.28	45.28	\$29
$J_2(2)$	66.53	68.11	65.18	68.11	\$24
$J_2(3)$	109.29	118.3	115.71	118.3	\$29
$J_2(4)$	158.56	153.59	155.61	158.56	\$24
$J_2(5)$	208.54	192.21	192	208.54	\$19

Table B.4 Transition Probability Matrix

Price Level 1		S_{k+1}					
	States	0	1	2	3	4	5
S_k	0	1	0	0	0	0	0
	1	0.5	0.5	0	0	0	0
	2	0.2	0.4	0.4	0	0	0
	3	0.1	0.4	0.4	0.1	0	0
	4	0.1	0.2	0.5	0.1	0.1	0
	5	0.1	0.1	0.3	0.4	0.05	0.05

Price Level 2		S_{k+1}					
	States	0	1	2	3	4	5
S_k	0	1	0	0	0	0	0
	1	0.6	0.4	0	0	0	0
	2	0.2	0.3	0.5	0	0	0
	3	0.1	0.5	0.3	0.1	0	0
	4	0.1	0.25	0.55	0.1	0.1	0
	5	0.1	0.15	0.25	0.4	0.05	0.05

Price Level 3		S_{k+1}					
	States	0	1	2	3	4	5
S_k	0	1	0	0	0	0	0
	1	0.7	0.3	0	0	0	0
	2	0.3	0.4	0.3	0	0	0
	3	0.1	0.35	0.5	0.05	0	0
	4	0.1	0.25	0.5	0.05	0.1	0
	5	0.1	0.15	0.3	0.35	0.03	0.07